Numerical Analysis Preliminary Examination questions
August 2016

<u>Instructions</u>: NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Calculate the following sum by hand in IEEE double precision computer arithmetic, using the Rounding to Nearest Rule:

$$(1 + (2^{-51} + 2^{-52} + 2^{-60})) - 1.$$

2. (a) Show how to calculate $\sqrt{5}$ using Newton's method.

   (b) Perform two iterations of Newton's method starting with an initial guess 2.

   (c) Prove that the Newton's method generates a sequence that converges to $\sqrt{5}$ with a quadratic rate.

3. (a) Find the solution to the following boundary value problem (BVP)

$$y'' = 2,\ 0 < t < 1,\ y(0) = 0,\ y(1) = 1.$$

   using a numerical method of your choice with 3 mesh points 0, 0.5 and 1.

   (b) Find the analytical solution to the BVP.

   (c) Find the maximum error between the analytical and numerical solutions on $[0, 1]$.

4. (a) Write down the Lagrange interpolating polynomial $P(t)$ for the points $(0, 3), (1, 7), (2, 5)$.

   (b) Set up a linear system of equations $A\vec{x} = \vec{b}$ which determines the coefficients of the above interpolating polynomial in the monomial basis $\{1, t, t^2\}$.

   (c) Compute the LU-decomposition of the matrix A using partial pivoting.

   (d) Write down the formula for the Householder transformation to eliminate the first column of A.

5. Consider the QR Iteration algorithm for a matrix $A = A_0$

   for k=1:m

   Compute the QR factorization: $Q_k R_k = A_{k-1}$

   Set $A_k = R_k Q_k$

   end

   (a) Show that the matrices A_k all have the same eigenvalues.

   (b) Assume that A is a real symmetric matrix with distinct eigenvalues and that A_m is equal to a diagonal matrix Λ. Describe how to determine the eigenvectors of A from the algorithm.

6. A quadrature rule is given by:

$$\int_{-1}^{1} f(x)\, dx = w_1 f(-a) + w_2 f(0) + w_3 f(a)$$

   (a) Find w_1, w_2, w_3, a that will make this quadrature rule exact for all polynomials of as high a degree as possible.

   (b) Use this quadrature rule to explain how to numerically compute $\displaystyle\int_0^1 \frac{1}{x+1}\, dx$.

7. Consider the initial value problem

$$\frac{dy}{dt} = a + by(t) + c\cos y(t), \qquad 0 \le t \le 1$$

   where $y(0) = 1$ and $a, b, c > 0$ are constants. Let us suppose that the solution satisfies $\max\limits_{0\le t\le 1} |y''(t)| = M < \infty$. Consider the approximation $y_{k+1} = y_k + (a + by_k + c\cos(y_k))h$ for $k = 0, 1, 2, \ldots, N-1$ and let $y_0 = y(0)$ and $h = \dfrac{1}{N}$. Prove that

$$|y(1) - y_N| \le \frac{Mhe^{b+c}}{2(b+c)}.$$