Numerical Analysis – Preliminary Exam
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AUGUST 2011

Instructions: Answer all four of the following problems. This is a closed book, closed notes exam. Show all of your work for full credit. No calculators or computers are allowed.

1. (a) Consider the fixed point iteration scheme used to solve $f(x) = 0$ (i.e. find α such that $f(\alpha) = 0$)

$$x_{n+1} = g(x_n) = x_n - f(x_n)/c,$$

where c is a constant value. Under what conditions on the value of c will this scheme be locally convergent? What will the convergence rate be in general? Is there any value of c for which the scheme will converge quadratically? If so, show details.

(b) Show that Newton's method for solving $f(x) = 0$ corresponds to a special case of fixed point iteration (i.e. identify the corresponding function $g(x)$ where $x_{n+1} = g(x_n)$). Show that if $f'(\alpha) \neq 0$ the convergence is locally quadratic.

(c) Solve $x^2 = 0$ with Newton's Method (start with the guess $x_0 = 1$ and determine an explicit formula for the k^{th} iterate) and discuss the convergence properties in this case.

2. Derive the maximum time step allowed for numerical stability of Euler's method applied to the initial value problem stated below.

$$\begin{aligned} \frac{dy}{dt} &= -5y, \\ y(0) &= 1, \end{aligned}$$

Is there a numerical stability requirement if the Backward Euler method were used? Explain why or why not.

3. (a) Suppose that you have at your disposal your favorite algorithm to solve a nonsingular linear system of equations. Describe a numerical procedure based on the finite difference method for solving the following boundary value problem

$$\begin{aligned} u'' + u &= 0, \\ u(0) &= 0, \\ u(1) &= 1. \end{aligned}$$

that could make use of your linear solver (ignore for now that it has an analytical solution). Use a uniform grid with $N+1$ equally-spaced points with $x_j = (j-1)/N$ for $j = 1, \ldots, N+1$.

(b) Now suppose that instead of a linear solver you have at your disposal your favorite initial value problem solver (that solves a system of first order differential equations, e.g. Euler's method, Runge-Kutta, etc.) and your favorite nonlinear solver (e.g. bisection, Newton's method, secant method, ...). Outline the details of a procedure that you could use in this case to solve the above boundary value problem. Note that original problem involves a second order differential equation so be sure to provide details on the appropriate conversion required. Also, be sure to describe how you will obtain all the necessary ingredients for the nonlinear solver you choose (e.g. function values, derivatives, Jacobian, ...)

(c) Finally, suppose that you wanted to solve the following eigenvalue problem numerically (ignore for now that it has a relatively easy to find analytical solution) to obtain the three smallest positive eigenvalues λ and corresponding eigenfunctions u

$$\begin{aligned} u'' + \lambda u &= 0, \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned}$$

If you had at your disposal your favorite eigenvalue solver (i.e. a numerical algorithm that, given a matrix A, finds the eigenvalues λ and eigenvectors x

of $Ax = \lambda x$) describe how you could make use of this to solve the differential eigenvalue problem. If you did not have an eigenvalue solver but had any of your other favorite algorithms listed above describe how you could solve the differential eigenvalue problem (again ignoring an analytical solution).

4. Consider the linear least squares problem where you wish to find $\boldsymbol{x}$ such that

$$||\boldsymbol{b} - A\boldsymbol{x}||_2^2,$$

is minimized. Here A is a full rank $m \times n$ matrix ($m > n$), $\boldsymbol{b}$ is a given vector of length m and $\boldsymbol{x}$ is a solution vector of length n.

(a) Suppose A has the QR factorization $A = QR$ where Q is an $m \times m$ orthogonal matrix and R is an $m \times n$ matrix with the upper $n \times n$ portion of R upper triangular and the remaining portion of R zero. Show how the solution of the least squares problem can be found by solving an $n \times n$ triangular system that makes use of Q and/or R.

(b) Consider the QR factorization of the matrix given below

$$A = \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} = QR = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 15\sqrt{3} \\ 0 & 5\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

Describe in words, generally but not doing specific computations, two ways in which the QR factorization of a matrix could be obtained.