Math 685: Preliminary Exam – Fall 2013

August 21

Instructions: This is a closed book, closed notes in class exam. You are required to do any four out of five problems but are strongly encouraged to attempt all five of them.

1. (*a*) *Projections*: Let H be a linear space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let S be a subspace of H, and let $u \in H$. Prove that $u^* \in S$ is the best approximation of u within S if and only if u^* is the orthogonal projection of u onto S that is

$$\langle u - u^*, v \rangle = 0 \quad \forall v \in S.$$

Hint: argue with the function $f(t) = \left\|u - (u^* + tv)\right\|^2$.

(*b*) *Bessel Inequality*: Let $u \in L^2(a,b) := \left\{ f : \int_a^b \left|f(x)\right|^2 dx < \infty \right\}$. Let $\{p_i\}_{i=1}^n$ be a set of orthonormal functions in $L^2(a,b)$ and $S := \mathsf{span}\,\{p_i\}_{i=1}^n$. Let $u_n = \sum_{i=1}^n \alpha_i p_i \in S$ be the best (least squares) approximation of u. Show that $\|u - u_n\|^2 = \|u\|^2 - \|u_n\|^2$ and conclude

$$\sum_{i=1}^n |\alpha_i|^2 \le \|u\|^2 .$$

2. *Gaussian Quadrature*. Derive the two point Gaussian integration formula $G_2(f)$ for

$$I(f) = \int_a^b f(x)dx.$$

Indicate for what polynomial degree this rule is exact.

3. Let $b \in \mathbb{R}^n$ be given and $A \in \mathbb{R}^{n \times n}$ be a strictly diagonally dominant matrix, namely,

$$\max_{1 \le i \le n} \sum_{j \ne i} \frac{|a_{ij}|}{|a_{ii}|} = r < 1.$$

Show that the *Jacobi* iteration $x_{k+1} = D^{-1}(L+U)\,x_k + D^{-1}b$ converges regardless of the initial guess $x_0 \in \mathbb{R}^n$.

Hint: Recall, $A = D - L - U$, where D is diagonal, L is lower triangular and U is upper triangular.

4. The following MATLAB script can be used to compute its eigenvalues of $A \in \mathbb{R}^{n \times n}$:

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B = A;
for k = 1:m
    [Q, R] = qr(B);
    B      = RQ;
end
```

(*a*) Show that consecutive iterates B are similar, and deduce they are all similar to A (Recall that two square matrices B, C are similar if $C = SBS^{-1}$).

(*b*) Show that A and B have the same eigenvalues (do not use any theorems but rather prove this fact directly).

(*c*) Are the eigenvectors of A and B the same? If not, find a relation among them.

5. Consider the second order elliptic partial differential equation

$$-\text{div }(\alpha(x)\text{grad } u) = f \tag{1}$$

posed on a smooth domain $\Omega \subset \mathbb{R}^2$, subject to Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

(a) Define $H_0^1(\Omega)$, and state a weak formulation of the problem (1). This should have the form

$$\text{find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V, \tag{2}$$

where $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are bilinear forms and $V = H_0^1(\Omega)$.

(b) Identify sufficient conditions on α that guarantee that

$$a(u, u) \geq \gamma\|u\|_V^2, \quad a(u, v) \leq \Gamma\|u\|_V\|v\|_V$$

for all $u, v \in V$, where $\|\cdot\|_V$ is a suitable norm defined on V.

(c) Let V_h denote a finite element subspace of V. Define the discrete weak solution of (1) on V_h and show that if u is the solution to (2), then there exists a positive constant $C(\gamma, \Gamma)$ such that

$$\|u - u_h\|_V \leq C(\gamma, \Gamma) \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{3}$$