Numerical Analysis – Preliminary Exam
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Instructions: Answer any **four** of the following **seven** problems. **Clearly indicate which four are to be graded**. This is a closed book, closed notes exam. Show all of your work for full credit. No calculators or computers are allowed.

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1. Let $x_i^*,\ i \geq 0$ denote positive numbers on a computer. With a unit round-off error δ, $x_i^* = fl(x_i) = x_i\ (1 + \epsilon_i)$ with $|\epsilon_i| \leq \delta$, where x_i are positive exact numbers.

   (a) Consider the product $P_n = \prod_{i=0}^{n} x_i$. Let the floating point approximation $fl(P_n) = P_n^* = P_n(1 + \epsilon)$. Prove that ϵ is bounded as $\epsilon \leq e^{\delta(2n+1)} - 1$.

   (b) Consider the expression $S = x_1 \cdot x_2 \cdot x_3 + x_4 \cdot x_5 \cdot x_6$. Let S^* be the floating point approximation of S. Prove that $\dfrac{S^*}{S} \leq e^{6\delta}$.

2. (a) Work out by hand, showing all necessary details and steps, the first Newton iterate x_1 for solving the problem $f(x) = 0$ where $f(x) = 1 - x - \ln x$ given an initial guess $x_0 = 1/2$.

   (b) Describe the convergence properties of Newton's method (to the exact root $x = 0$) applied to the function $f(x) = x^m$ where m is a positive integer. Be as specific as possible in terms of computing convergence rates.

   (c) Consider the nonlinear system $\vec{f}(\vec{x}) = 0$ where $\vec{f} = (f_1, f_2)$, $\vec{x} = (x, y)$ and

   $$\begin{aligned} f_1(x, y) &= y - x^2 - c, \\ f_2(x, y) &= x^2 + y^2 - 1, \end{aligned}$$

   where c is a constant to be specified.

   For the case $c = 0$ write out by hand the specific linear system that must be solved in order to obtain the first Newton iterate $\vec{x}_1$ given the starting vector $\vec{x}_0 = (1/2, 1/2)$. Discuss convergence rates for this method and describe how these rates could be influenced by the value of the parameter c.

3. Suppose that a function g maps the interval $[a, b]$ into itself and g satisfies a Lipschitz condition with a Lipshitz constant $0 \leq \lambda < 1$, then

   (a) Show that the sequence $x_{n+1} = g(x_n), n = 0, 1, \ldots$ (for any initial approximation $x_0 \in [a, b]$) converges to a unique fixed point ξ in $[a, b]$.

   (b) Moreover, prove the following error bound involved in using the sequence x_n to approximate ξ given by: $|x_n - \xi| \leq \dfrac{\lambda^n}{1 - \lambda}|x_1 - x_0| \ \ \forall n \geq 1$.

4. (a) Derive the two point Gaussian integration formula $G_2(f)$ for $I(f) = \int_{-1}^{1} f(x)dx$.

   (b) Indicate for what polynomial degree this rule is exact. Use the the two-point Gauss Quadrature fourmula derived to evaluate the integral $\int_1^4 \frac{dx}{5x+1}$.

5. (a) Let f'' be continuous in $[a,b]$ and let $a = t_0 < t_1 < \ldots < t_n = b$. If S is a complete cubic spline interpolating f at the knots t_i for $0 \le i \le n$, then prove that,

$$\int_a^b \left[S''(x)\right]^2 \, dx \;\le\; \int_a^b \left[f''(x)\right]^2 \, dx$$

   (b) A complete cubic spline S for a function f is defined on $[1,3]$ by,

$$s(x) = \begin{cases} s_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if} \quad 1 \le x < 2 \\ s_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3 & \text{if} \quad 2 \le x \le 3 \end{cases}$$

   Given $f'(1) = f'(3)$, find a, b, c, d.

6. Determine the total number of operations (addition, subtraction, multiplication, division) for the Gaussian Elimination algorithm described below:

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input $n, (a_{ij})$

for $k = 1$ to $n-1$ do

  for $i = k+1$ to n do

    $z = \dfrac{a_{ik}}{a_{kk}}$

      for $j = k$ to n do

        $a_{ij} = a_{ij} - z a_{kj}$

      end do

  end do

end do

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7. Consider the initial-value problem $y'(t) = f(t,y)$ for $0 \le t \le 1$ with $y(0) = a$. Consider the one-step method

$$y_{k+1} = y_k + h\ \Phi(t_k, y_k, h)$$

with $y_0 = a$, $h = 1/N$, and $t_k = kh$ for $k = 0, 1, \cdots, N$. Assume that there is a constant L such that $|\Phi(t,y,h) - \Phi(t,z,h)| \le L\ |y - z|$ for all $t, y, z \in \mathcal{R}$. Furthermore, assume that the solution $y(t)$ satisfies $|y(t+h) - y(t) - h\Phi(t, y(t), h)| \le c\ h^{p+1}$ for all $t, h \in [0,1]$. Prove that

$$|y_N - y(1)| \le c\,\frac{h^p}{L}\,(e^L - 1).$$