Math 685: Preliminary Exam – Spring 2014

JANUARY 15

Instructions: This is a closed book, closed notes in class exam. You are required to do any four out of five problems but are strongly encouraged to attempt all five of them.

1. Suppose you have a large set of data (x_j, f_j), $j = 0, 1, \ldots, n$, available at equidistant points $x_j = \frac{j}{n}$ on the interval $[0, 1]$. Suppose you want to approximate these data with a polynomial of degree k where $1 \ll k \ll n$.

   (a) Formulate a least squares problem for the polynomial fit of the form $Ay = b$. Explain how you will define the entries of the matrix A and the vector b.

   (b) Derive a formula for the solution of the least squares problem.

2. *Gaussian Quadrature.*

   (a) Derive the two point Gaussian integration formula $G_2(f)$ for

   $$I(f) = \int_{-1}^{1} f(x)dx.$$

   Indicate for what polynomial degree this rule is exact.

   (b) Evaluate the integral using the two-point Gauss Quadrature fourmula derived:

   $$\int_{1}^{4} \frac{dx}{x+2}.$$

3. Suppose you need to minimize $f(x) = \frac{1}{2}x^T Ax - b^T x + c$, $x \in \mathbb{R}^n$, where A is symmetric positive definite.

   (a) Let x_0 be the starting point. Perform one step of the steepest descent method with exact line search and find the expression for x_1.

   (b) Prove that the function $f(x)$ is minimized when $Ax = b$.

   (c) Suppose now that you wish to solve $Ax = b$ using an iterative method. Let x_0 represent the starting point and obtain an expression based on the Jacobi method for the first iterate x_1.

   (d) Compare and contrast in general terms the Jacobi method and the Gauss-Seidel iterative methods.

4. Given a matrix $S = [s_1, \ldots, s_N] \in \mathbb{R}^{M \times N}$ of rank $K \leq N$ and $M \geq N$, the *proper orthogonal decomposition (POD)* is a method to approximate S with $d \ll K$ linearly independent vectors in $\text{span}\{s_i\}_{i=1}^{N}$. Let $S = U\Sigma V^T$ be the singular value decomposition of S, i.e.

   $$U = [u_1, \ldots, u_M] \in \mathbb{R}^{M \times M}, \quad V = [v_1, \ldots, v_N] \in \mathbb{R}^{N \times N},$$

   are orthogonal matrices and $\Sigma = \begin{bmatrix} D \\ 0 \end{bmatrix} \in \mathbb{R}^{M \times N}$, with diagonal matrix $D = \text{diag}(\sigma_1, \ldots, \sigma_N)$ containing the singular values of S:

   $$\sigma_1 \geq \cdots \geq \sigma_K > \sigma_{K+1} = \cdots = \sigma_N = 0.$$

(a) Show that $\text{span}\{s_j\}_{j=1}^N = \text{span}\{u_j\}_{j=1}^K$.

(b) Let $A_i = u_i v_i^T$ be a rank-1 matrix for $1 \le i \le N$. Show the orthogonal decomposition $S = \sum_{i=1}^K \sigma_i A_i$ in the Frobenius inner product

$$\langle A, B\rangle_F = \sum_{i=1}^M \sum_{j=1}^N a_{ij} b_{ij},$$

i.e. show that $\langle A_i, A_j\rangle_F = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

(c) For $d < K$ we may approximate S with $\sum_{i=1}^d \sigma_i A_i$; this is the POD of S. Show the error formula

$$\left\| S - \sum_{i=1}^d \sigma_i A_i \right\|_F^2 = \sum_{i=d+1}^K \sigma_i^2,$$

where $\|\cdot\|_F$ is the norm subordinate to the Frobenius inner product. This expression could be used to find the number d so that the error $\sum_{i=d+1}^K \sigma_i^2$ is within a given tolerance.

5. Consider the two-point boundary value problem

$$\mathcal{A}u := -(a(x)u')' + c(x)u = f(x) \quad \text{in } (0,1), \quad u(0) = u(1) = 0,$$

where $a(x) \ge a_0 > 0$ and $c(x) \ge 0$ are smooth functions in $[0,1]$, and $f(x) \in L^2(0,1)$.

(a) Derive the variational form for this problem.

(b) Set $a(x) = 1$ and $c(x) = 0$ and consider a partition of $[0,1]$

$$0 = x_0 < x_1 < \cdots < x_N = 1.$$

Write the system of equations produced by the finite element method with continuous piecewise linear elements $\Phi_j(x)$, $j = 1, 2, \ldots, N-1$,

$$\Phi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x_{j-1} \le x \le x_j, \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Show that this system of equations is equivalent to the one produced by the finite difference method for the uniform partition of $f \equiv$ constant.