Department of Mathematical Sciences George Mason University

Linear Analysis Preliminary Exam, January 2011

This exam consists of 4 questions.

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(1) Let (X, d) be a complete metric space, and for any set $A \subset X$ define its diameter by

$$\mathrm{diam}(A) = \sup\{d(x, y) \; : \; x, y \in A\} \, .$$

Let $(A_n)_{n\in\mathbb{N}}$ denote a nested sequence of closed nonempty subsets of X such that $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$.

(a) Show that there exists an $x \in X$ which is contained in all sets A_n.

(b) Does the result of (a) still hold if the completeness of X is replaced by the compactness of X?

(c) Does the result of (a) still hold if the assumption $\lim_{n\to\infty} \mathrm{diam}(A_n) = 0$ is removed?

(2) Let X denote a linear space.

(a) Assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Suppose further that there exists a constant $C > 0$ such that

$$\|x\|_1 \le C\|x\|_2 \quad \text{for all} \quad x \in X \, . \tag{1}$$

Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are then already equivalent.

(b) On the space $X = C[0, 1]$ of continuous functions consider the two norms

$$\|x\|_1 = \int_0^1 |x(t)| \; dt \quad \text{and} \quad \|x\|_2 = \max_{t\in[0,1]} |x(t)| \; .$$

Show that the norms satisfy (1), but that they are not equivalent. How can this be reconciled with (a)?

(3) Let X denote a Banach space, and Y a normed linear space. Furthermore, assume that $T \in \mathcal{L}(X, Y)$ is a bounded linear map which is one-to-one, i.e., the inverse $T^{-1} : R(T) \to X$ exists, where $R(T)$ denotes the range of T (which does not have to be all of Y).

(a) Show that if T^{-1} is continuous, then the range $R(T)$ is closed.

(b) Consider the Banach spaces $X = Y = C[0, 1]$ equipped with the maximum norm and let $T : X \to Y$ be defined as

$$(Tx)(t) = \int_0^t x(s) \; ds \, .$$

Show that T is one-to-one and continuous, find its norm $\|T\|$, and determine the range $R(T)$. Is the range closed? Is the inverse operator T^{-1} continuous? (Justify your answers!)

(4) Let H denote a Hilbert space, let $T \in \mathcal{L}(H, H)$ denote a bounded linear operator, and let $T^* \in \mathcal{L}(H, H)$ denote its adjoint.

(a) Prove that $\|T\| = \|T^*\|$.

(b) Consider specifically the case $H = \ell_2$ and the operator T defined as

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots) \quad \text{for all} \quad x = (x_1, x_2, x_3, \dots) \in \ell_2 \, .$$

Find the adjoint T^* and determine the norms $\|T\|$ and $\|T^*\|$ directly.
