Linear Analysis Preliminary Exam, January 2011

This exam consists of 4 questions.

(1) Let \((X, d)\) be a complete metric space, and for any set \(A \subset X\) define its diameter by
\[
\text{diam}(A) = \sup \{d(x, y) : x, y \in A\}.
\]
Let \((A_n)_{n \in \mathbb{N}}\) denote a nested sequence of closed nonempty subsets of \(X\) such that \(\lim_{n \to \infty} \text{diam}(A_n) = 0\).

(a) Show that there exists an \(x \in X\) which is contained in all sets \(A_n\).
(b) Does the result of (a) still hold if the completeness of \(X\) is replaced by the compactness of \(X\)?
(c) Does the result of (a) still hold if the assumption \(\lim_{n \to \infty} \text{diam}(A_n) = 0\) is removed?

(2) Let \(X\) denote a linear space.

(a) Assume that \(\|\cdot\|_1\) and \(\|\cdot\|_2\) are two norms on \(X\) such that \((X, \|\cdot\|_1)\) and \((X, \|\cdot\|_2)\) are Banach spaces. Suppose further that there exists a constant \(C > 0\) such that
\[
\|x\|_1 \leq C\|x\|_2 \quad \text{for all} \quad x \in X.
\]
Show that the norms \(\|\cdot\|_1\) and \(\|\cdot\|_2\) are then already equivalent.
(b) On the space \(X = C[0, 1]\) of continuous functions consider the two norms
\[
\|x\|_1 = \int_0^1 |x(t)| \, dt \quad \text{and} \quad \|x\|_2 = \max_{t \in [0,1]} |x(t)|.
\]
Show that the norms satisfy (1), but that they are not equivalent. How can this be reconciled with (a)?

(3) Let \(X\) denote a Banach space, and \(Y\) a normed linear space. Furthermore, assume that \(T \in \mathcal{L}(X, Y)\) is a bounded linear map which is one-to-one, i.e., the inverse \(T^{-1} : R(T) \to X\) exists, where \(R(T)\) denotes the range of \(T\) (which does not have to be all of \(Y\)).

(a) Show that if \(T^{-1}\) is continuous, then the range \(R(T)\) is closed.
(b) Consider the Banach spaces \(X = Y = C[0, 1]\) equipped with the maximum norm and let \(T : X \to Y\) be defined as
\[
(Tx)(t) = \int_0^t x(s) \, ds.
\]
Show that \(T\) is one-to-one and continuous, find its norm \(\|T\|\), and determine the range \(R(T)\). Is the range closed? Is the inverse operator \(T^{-1}\) continuous? (Justify your answers!)

(4) Let \(H\) denote a Hilbert space, let \(T \in \mathcal{L}(H, H)\) denote a bounded linear operator, and let \(T^* \in \mathcal{L}(H, H)\) denote its adjoint.

(a) Prove that \(\|T\| = \|T^*\|\).
(b) Consider specifically the case \(H = \ell_2\) and the operator \(T\) defined as
\[
T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots) \quad \text{for all} \quad x = (x_1, x_2, x_3, \ldots) \in \ell_2.
\]
Find the adjoint \(T^*\) and determine the norms \(\|T\|\) and \(\|T^*\|\) directly.