Preliminary Exam, Linear Analysis, January 2010

State any theorems that you use.

1. A subset $S$ of a metric space $(X, d)$ is called perfect if it is closed and every point of $S$ is an accumulation point of $S$. Prove that if $S$ is a nonempty perfect subset of a complete metric space, then $S$ is uncountable. (Hint: Assume $S$ is perfect and countable. Use Baire’s theorem or the Nested sphere theorem to deduce a contradiction.)

2. Let $X = C[0, 1]$ be the space of continuous function defined on $[0, 1]$, with the norm $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$. For any $f \in X$, define $T(f) = f(1) - f(0)$.

(a) Prove that $T$ is a continuous linear functional on $X$.

(b) Find the norm of $T$.

3. Let $E$ be a Banach space, and let $I$ be the identity operator on $E$. Suppose $A$ is a bounded linear operator mapping $E$ into itself, such that $\|A\| < 1$. Define $S = I - A$.

(a) Show that $S$ is invertible and $S^{-1} = \sum_{n=0}^{\infty} A^n$.

(b) Prove that $\|S^{-1}\| \leq (1 - \|A\|)^{-1}$.

4. Let $l_2$ be the linear space consisting of all infinite sequences $v = (x_1, x_2, \ldots)$ of real numbers $x_1, x_2, \ldots$ satisfying the convergence condition $\sum_{k=1}^{\infty} x_k^2 < \infty$, equipped with the norm $\|v\| = \left(\sum_{k=1}^{\infty} x_k^2\right)^{1/2}$.

Define the linear operator $A$ mapping $l_2$ into itself by $A(x_1, x_2, x_3, \ldots) = (x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots)$.

Show that $A$ is bounded and injective, but is not bounded below; that is, that we can find a sequence of vectors $\{v_n\}$ in $l_2$ with $\|v_n\| = 1$ for all $n \geq 1$ and $\|Av_n\| \to 0$ as $n \to \infty$. (Hint: For each $n \geq 1$ consider the sequence whose first $n$ terms equal $1/\sqrt{n}$ and the rest of terms are zero.)