Preliminary Exam, Linear Analysis, January 2010

State any theorems that you use.

1. A subset S of a metric space (X, d) is called *perfect* if it is closed and every point of S is an accumulation point of S. Prove that if S is a nonempty perfect subset of a complete metric space, then S is uncountable. (Hint: Assume S is perfect and countable. Use Baire's theorem or the Nested sphere theorem to deduce a contradiction.)

2. Let $X = C[0, 1]$ be the space of continuous function defined on $[0, 1]$, with the norm $\|f\| = \max_{0\leq t\leq 1} |f(t)|$. For any $f \in X$, define
$$T(f) = f(1) - f(0).$$
(a) Prove that T is a continuous linear functional on X.
(b) Find the norm of T.

3. Let E be a Banach space, and let I be the identity operator on E. Suppose A is a bounded linear operator mapping E into itself, such that $\|A\| < 1$. Define $S = I - A$.
(a) Show that S is invertible and
$$S^{-1} = \sum_{n=0}^{\infty} A^n.$$
(b) Prove that $\|S^{-1}\| \leq (1 - \|A\|)^{-1}$.

4. Let l_2 be the linear space consisting of all infinite sequences $v = (x_1, x_2, \ldots)$ of real numbers $x_1, x_2, \ldots$ satisfying the convergence condition $\sum_{k=1}^{\infty} x_k^2 < \infty$, equipped with the norm
$$\|v\| = \Big(\sum_{k=1}^{\infty} x_k^2\Big)^{1/2}.$$
Define the linear operator A mapping l_2 into itself by
$$A(x_1,\, x_2,\, x_3,\, \ldots) = (x_2 - x_1,\, x_3 - x_2,\, x_4 - x_3,\, \ldots).$$
Show that A is bounded and injective, but is not bounded below; that is, that we can find a sequence of vectors $\{v_n\}$ in l_2 with $\|v_n\| = 1$ for all $n \geq 1$ and $\|Av_n\| \to 0$ as $n \to \infty$. (Hint: For each $n \geq 1$ consider the sequence whose first n terms equal $1/\sqrt{n}$ and the rest of terms are zero.)