Linear Analysis Preliminary Exam, August 2010

This exam consists of four questions.

State any theorem that you use.

1. Let A and B be disjoint closed subsets of a metric space (X, d). Give a direct proof for the existence of disjoint open subsets U_A and U_B of X such that $A \subset U_A$ and $B \subset U_B$ (do not use a reference on normality of a metric space).

2. Let $X = C[0,1]$ be the space of real-valued continuous function on $[0,1]$, with the norm $\|f\| = \max_{0 \le t \le 1} |f(t)|$. For any $f \in X$, define

$$T(f) = \int_0^1 x f(x)\, dx.$$

   Prove that T is a bounded linear operator on X and find its norm.

3. Given a Banach space E, let

$$B_{\rho_n}(z_n) = \{x \in E : \|x - z_n\| \le \rho_n\},\ z_n \in E,\ \rho_n \ge 0,\ n = 1, 2, \ldots,$$

   be a sequence of closed balls in E such that $B_{\rho_1}(z_1) \supseteq B_{\rho_2}(z_2) \supseteq \ldots$. Prove that $\bigcap_n B_{\rho_n}(z_n) \ne \varnothing$. (Hint: if a ball $B_\delta(x)$ lies in a ball $B_\rho(z)$, then $\delta \le \rho$ and $\|x - z\| \le \rho - \delta$. It is not assumed that ρ_n tends to 0 as $n \to \infty$.)

4. Let $\{x_1, x_2, \ldots\}$ be an orthonormal basis for a Hilbert space H, and let $\{z_1, z_2, \ldots\}$ be an orthonormal set in H such that

$$\sum_{n=1}^{\infty} \|x_n - z_n\|^2 < 1.$$

   Prove that $\{z_1, z_2, \ldots\}$ is an orthonormal basis for H.