Department of Mathematical Sciences
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Linear Analysis Preliminary Exam - August 2014

Instructions: This exam consists of four problems. You are to do all four of the problems. Closed books, closed notes. State any theorems that you use.

1. Let H be a Hilbert space. Let H_1 be a closed subspace of H. Let $P : H \to H$ be the orthogonal projection operator of H onto H_1. Let $A : H \to H$ be a continuous linear operator. Recall that in this setting, the adjoint $A^* : H \to H$ is defined as the operator such that for all $x \in H$ and $y \in H$, $(x, A^*y) = (Ax, y)$.

   (a) Is the projection operator P bounded? If so, what is its norm? Is P self-adjoint? Justify all of your answers.

   (b) Assume A is bounded. Define the operator $A_1 = PAP$. Show that A_1 is bounded. Find and prove an inequality relating the norm of A_1 to that of A.

   (c) Assume is A is self-adjoint and $A_1 = PAP$ as in (b). Is A_1 self-adjoint? If you believe so, prove it; if you believe not, find a counterexample to disprove it.

2. Let X be an inner product space, and let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal set. Recall that $\{u_k\}$ is defined to be complete if the closure of the span of $\{u_k\}$ contains X.

   (a) If X is complete and $\{u_k\}$ is complete, show that there is no nonzero element $x \in X$ such that $(x, u_k) = 0$ for all k. (Hint: Use the Riesz-Fisher Theorem.)

   (b) If X is complete but $\{u_k\}$ is not complete, show that there is a nonzero element $x_* \in X$ such that $(x_*, u_k) = 0$ for all k.

3. Let $K : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous function. For each $x \in C[0, 1]$, and for $t \in [0, 1]$, define a function $F(x) \in C[0, 1]$ as follows

$$(F(x)(t) = \int_0^1 K(t, s)x(s)\, ds.$$

   (a) Using the $\epsilon - \delta$ definition of continuity, show that $F : C[0, 1] \to C[0, 1]$ is continuous with respect to the metric d_∞.

   (b) For $K(t, s) = t$, find $\|F\|$. That is, find a value M such that (i) $\|F(x)\|_\infty \leq M\|x\|_\infty$ for all $x \in C[0, 1]$, and (ii) there is a function $x \in C[0, 1]$ such that $\|F(x)\|_\infty = M\|x\|_\infty$.

4. Let X be a complete metric space with metric d. Let $A : X \to X$ be a mapping. Assume that there is a constant $a < 1$ such that for every $x, y \in X$, $d(Ax, Ay) < ad(x, y)$.

   (a) Show that A is continuous.

   (b) For any point $x_0 \in X$, define $x_1 = Ax_0$, and recursively, $x_n = Ax_{n-1}$. Show that for all positive integers $n < m$, $d(x_n, x_m) \leq a^n/(1 - a)d(x_0, x_1)$.

   (c) Using part (b), show that there is a unique point x_* such that $Ax_* = x_*$.