Linear Analysis Preliminary Exam January 2015

(1) Let X be a complex Hilbert space, and let $T \in \mathcal{L}(X,X)$ denote the orthogonal projection onto a closed subspace $M \subseteq X$.

(a) Determine the kernel $N(T-\lambda I)$ and the range $R(T-\lambda I)$ of $T-\lambda I$ for each $\lambda \in \mathbb{C}$.

(b) Find all $\lambda \in \mathbb{C}$ for which $T-\lambda I$ has an inverse $(T-\lambda I)^{-1}$ in $\mathcal{L}(X,X)$. Justify your answer.

(c) What is the spectrum of $T-\lambda I$?

(2) (a) State and prove Hölder's Inequality for a pair of vectors in $\mathbb{R}^n$.

(b) Under what conditions does it become an equality? Justify your answer.

(c) If $\mathbb{R}^n$ is equipped with the norm

$$\|x\| = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

for some $p \geq 1$ what is the induced norm on the conjugate (dual) space?

(3) (a) Prove Banach's Contraction Mapping Principle, also known as a fixed point theorem or also as the method of successive approximation: every contraction mapping A defined on a complete metric space R has a unique fixed point.

(b) Let X be a Banach space and let $L : X \to X$ be a bounded linear operator. Show that for any given b, there is a unique solution to the equation

$$x = Lx + b$$

if $||L|| < 1$.

(c) Are there situations with $||L|| > 1$ for which there is still a unique solution to $x = Lx + b$? Explain your reasoning.

(4) Define the adjoint of a bounded linear operator A that maps one Banach space E into another Banach space E_1. Then prove that $||A^*|| = ||A||$, paying attention to the meanings implied in this relation.