Department of Mathematical Sciences
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Linear Analysis Preliminary Exam - January 2014

Instructions: This exam consists of four problems. You are to do all four of the problems. Closed books, closed notes. State any theorems that you use.

1. Define the function $f_p : \mathbb{R}^n \to \mathbb{R}$ for $n \geq 2$ by $f_p(x) = \sum_{k=1}^{n} |x_k|^p$.
   (a) Show that for $0 < p < 1$, $d_a(x, y) = (f_p(x - y))^{1/p}$ is not a metric on $\mathbb{R}^n$.
   (b) Show that for $0 < p < 1$, $d_b(x, y) = f_p(x - y)$ is a metric on $\mathbb{R}^n$.
   (c) For $p = 1/2$ and $n = 2$ find and graph the unit ball $\{x \in \mathbb{R}^2 : d_b(x, 0) \leq 1\}$.

2. Let X be the space $C[0, 1]$ under the norm $\| \cdot \|_p$ for $1 \leq p \leq \infty$.
   (b) Show that X is complete for $p = \infty$, but it is then not an inner product space.
   (a) Show that X is an inner product space for $p = 2$, but it is then not complete.

3. (a) State the Hahn-Banach Theorem.
   (b) Let X be a normed linear space, and let x_1 and x_2 be two points in X such that for every bounded linear functional f on X, $f(x_1) = f(x_2)$. Show that $x_1 = x_2$.
   (c) Let X be a normed linear space, and let $x_0 \in X$, where $x_0 \neq 0$. Then there exists a bounded linear functional f on X such that $f(x_0) = \|x_0\|$. For such an f, is it possible to make $\|f\| > 1$? Justify.

4. Consider the two linear spaces:

$$\ell^2 = \left\{ x = (x_1, x_2, \dots) : \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}$$

with norm $\|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}$, and

$$\ell^\infty = \left\{ x = (x_1, x_2, \dots) : \sup_k |x_k| < \infty \right\}.$$

with norm $\|x\|_\infty = \sup_k |x_k|$. Fix $a \in \ell^2$. Define the function T on ℓ^2 by

$$(Tx)_j = \sum_{k=1}^{j} a_k x_k.$$

   (a) Show that T is a continuous linear operator from ℓ^2 to ℓ^∞. Find the norm of T. Justify your answer.
   (b) Is T a continuous linear operator from ℓ^2 to ℓ^2? If so, find the norm. If not, prove that it is not.