Linear Analysis Preliminary Exam - January 2012

1. Let C, D be compact subsets of a normed linear space. Prove that the set
$$C + D = \{c + d : c \in C, d \in D\}$$
is compact.

2. Let X be a real normed linear space and $f : X \to \mathbb{R}$ a linear functional. Prove that the following are equivalent:
1. f is continuous.
2. f is continuous at 0.
3. There exists $M \geq 0$ such that $|f(x)| \leq M\|x\|$ for all $x \in X$.

3. Let X be a Banach space. A subset $S \subset X$ is called a Hamel basis of X if S is linearly independent and every element of X is a *finite* linear combination of elements of S. (i) Prove that if X is infinite dimensional, then S is uncountable. (You may use the fact that every finite dimensional linear subspace of X is closed.) (ii) Give an example of an infinite dimensional normed linear space which has a countable Hamel basis.

4. Let X be the space of real-valued continuous functions on $[0, 1]$ equipped with the norm
$$\|f\| = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$
Define the operator A on X by
$$Af(x) = \int_0^1 K(x, y) f(y) \, dy$$
where $K(x, y)$ is a function continuous on the square $[0, 1] \times [0, 1]$. Prove that (i) for each $f \in X$,Af is in X; (ii) $A : X \to X$ is a bounded linear operator with respect to the norm $\| \cdot \|$.