Linear Analysis Preliminary Exam August 2016

Instructions: This exam is closed book, closed notes, no calculator or other electronic device. Do all of the following four questions.

1. (a) Define a linearly independent set in a Hilbert space. Define an orthogonal set in a Hilbert space.

   (b) Prove that in a Hilbert space, an orthogonal set is a linearly independent set.

   (c) Given an example of a Hilbert space and a linearly independent set which is not an orthogonal set.

2. Let $C^1[a,b]$ denote the space of continuously differentiable functions under the norm $\|f\| = \max_{x\in[a,b]}\{|f(x)|, |f'(x)|\}$. Let $C[a,b]$ denote the space of continuous functions under the supremum norm.

   (a) Define a linear operator. Consider the mapping $\frac{d}{dx} : C^1[a,b] \to C[a,b]$. Prove that the mapping $\frac{d}{dx}$ is a linear operator on $C^1[a,b]$.

   (b) Define a completely continuous (compact) operator. Prove that the mapping $\frac{d}{dx}$ defined in (a) is a completely continuous (compact) operator. Hint: Use the following theorem:

   Theorem 1 (Arzela-Ascoli)**.** *For each n let $f_n(x)$ be a continuous function on the interval $[a,b] \subseteq \mathbb{R}$. If the sequence $\{f_n\}$ is both uniformly continuous and equicontinuous, then there is a subsequence $\{f_{n_k}\}$ which converges uniformly.*

3. (a) (i) Define a continuous linear functional on a normed linear space. (ii) Define the norm of a linear functional.

   (b) Let $C[a,b]$ denote the space of continuous functions under the supremum norm. Let $S : C[0,1] \to \mathbb{R}$ be a functional that assigns to each $f \in C[0,1]$ the real number

   $$Sf = \frac{f(0) + 4f(1/2) + f(1)}{6}.$$

   Show that S is a continuous linear functional. Find the norm of S.

4. Consider a linear map $A : X \to Y$ between two normed spaces X and Y.

   (a) (i) Define a bounded operator. (ii) Define a continuous operator. (iii) Prove that A is bounded if and only if A is continuous.

   (b) Suppose that Y is a Banach space and that A is continuous when restricted to some dense subset $D \subseteq X$. Prove that there is a unique continuous extension of A to X with the same norm.

   (c) Provide an example to show that the assumption that Y is a Banach space in (b) is indeed necessary.