Linear Analysis Preliminary Exam January 2016

Instructions: This exam is closed book, closed notes, no calculator or other electronic device. Do all of the following four questions.

1. Consider the linear space $X = C[0,1]$ equipped with the maximum norm $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let $r \in C[0,1]$ such that $r(t) > 0$ for $t \in [0,1]$. For $p \geq 1$ and $u \in C[0,1]$, define

$$\|u\|_{rp} = \left(\int_0^1 r(t)|u(t)|^p \, dt \right)^{1/p}.$$

   (a) Show that $\|u\|_{rp}$ defines a norm on X.

   (b) Show that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\|f_n - f\|_\infty \to 0$ for $n \to \infty$, then also $\|f_n - f\|_{rp} \to 0$.

   (c) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\|f_n - f\|_{rp} \to 0$ for $n \to \infty$, is it true that $\|f_n - f\|_\infty \to 0$? Justify your answer.

2. (a) State the Hahn-Banach Theorem.

   (b) Let X be a Banach space over the field K and let $M \subseteq X$. Assume that there exists $q \in X$ which is not in the closure of span(M). Show there is a continuous linear functional on X such that for every $m \in M$, $f(m) = 0$ and $f(q) = 1$.

   (c) Give an example of a Banach space X and a proper subspace M such that the only bounded linear functional on X which is zero on M is the zero function. Justify your answer.

3. Let X and Y be normed linear spaces. Let $T : X \to Y$ be an operator.

   (a) Show that T is a continuous linear operator if and only if T is a bounded linear operator.

   (b) Assume that T is continuous and that for all $x, y \in X$, $T(x + y) = T(x) + T(y)$. Show that T is a linear operator.

4. Let U and V be closed subspaces of a Hilbert space X, and let P_U and P_V denote the corresponding orthogonal projections.

   (a) Show that P_U is a bounded linear operator and find its norm.

   (b) Find the adjoint of P_U.

   (c) Show that $U \subseteq V$ if and only if $P_U = P_V P_U$.

   (d) When is P_U invertible? Justify.