Department of Mathematical Sciences George Mason University

Linear Analysis Preliminary Exam

This exam consists of 4 questions.

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1. Let (X,d) be a metric space. For a bounded set $S \subset X$, the diameter of S is defined as $\operatorname{diam}(S) = \sup\{d(x,y) : x,y \in S\}$. Prove, directly from the definitions, that a decreasing sequence of closed sets $C_n, n = 1,2,\ldots$ in a complete metric space has a nonempty intersection if $\lim_{n\to\infty} \operatorname{diam} C_n = 0$.

2. A function f is said to be continuously differentiable if both f and its derivative f' are continuous. Let X be the space of continuously differentiable real-valued functions on $[-1,1]$, with the norm $\|\cdot\|$ defined by

$$\|f\| = \|f\|_\infty + \|f'\|_\infty$$

where

$$\|g\|_\infty = \max\{|g(x)| : x \in [-1,1]\}.$$

Define a linear functional T on X as follows:

$$T(f) = f'(0)$$

Prove that T is bounded and find its norm. (For the second part, you may want to consider the functions $f_n(x) = \frac{1}{n}\sin(nx)$ for n large.)

3. Let l^2 be the Hilbert space of complex-valued sequences $\{x_n\}_{n=1}^\infty$ with the property that $\sum_{n=1}^\infty |x_n|^2 < \infty$, with norm given by $\|x\|^2 = \sum_{n=1}^\infty |x_n|^2$ and inner product given by $\langle x,y\rangle = \sum_{n=1}^\infty x_n\overline{y_n}$. Let $\lambda \in \mathbb{C}, 0 < |\lambda| < 1$, be fixed and define the operator T on l^2 by $Tx = \{\lambda^n x_n\}_{n=1}^\infty$.

   (a) Show that T is a bounded operator with $\|T\| = |\lambda|$.

   (b) Show that T is injective but that the operator $T^{-1}\colon \mathit{Range}(T) \to l^2$ is not bounded.

   (c) Show that T is not surjective, but that $\mathit{Range}(T)$ is dense in l^2.

4. Let X be the space of real-valued continuous functions on $[0,1]$ equipped with the norm

$$\|f\| = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}.$$

Define the operator A on X by

$$Af(x) = \int_0^1 K(x,y)f(y)\,dy$$

where $K(x,y)$ is a function continuous on the square $[0,1]\times[0,1]$. Prove that A is a bounded linear operator on X with respect to the norm $\|\cdot\|$.
