Linear Analysis Preliminary Exam

This exam consists of 4 questions.

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(1) Let (X, d) be a separable metric space, and Y any subspace of X. Prove that Y with the subspace topology is also separable.

(2) Let H be a (complex) Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Let $a \in H$ be a fixed element of H. Prove that $T(x) = \langle x, a \rangle$ is a continuous linear functional on H and find its norm. Is every continuous linear functional on H of this form?

(3) Consider the operator A defined on $L^2[0,1]$ by $Af(x) = \int_0^x f(t)\,dt$.

(a) Show that $Af \in C[0,1]$ whenever $f \in L^2[0,1]$.

(b) Prove that A is a bounded linear operator on $L^2[0,1]$ (where $L^2[0,1]$ is equipped with the usual norm $\|f\|_2^2 = \int_0^1 |f(x)|^2\,dx$). (Hint: You need not prove that A is linear.)

(c) Find A^*.

(4) Suppose that $\{x_n\}_{n=1}^{\infty}$ is an orthonormal system in a real Hilbert space H, x a vector in H and $\{a_n\}_{n=1}^{N}$ a finite sequence of real numbers.

(a) Show that

$$\left\| x - \sum_{n=1}^{N} a_n\, x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, x_n \rangle|^2 + \sum_{n=1}^{N} |a_n - \langle x, x_n \rangle|^2.$$

(b) Prove the following two identities as immediate consequences of the identity in part (a).

(i) $\displaystyle \left\| \sum_{n=1}^{N} a_n\, x_n \right\|^2 = \sum_{n=1}^{N} |a_n|^2$

(ii) $\displaystyle \left\| x - \sum_{n=1}^{N} \langle x, x_n \rangle\, x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, x_n \rangle|^2.$

(c) Explain why the result in part (a) is sometimes called the *Best Approximation Lemma*.
