Linear Analysis Preliminary Exam

This exam consists of 4 questions.

(1) Let \((X, d)\) be a separable metric space, and \(Y\) any subspace of \(X\). Prove that \(Y\) with the subspace topology is also separable.

(2) Let \(H\) be a (complex) Hilbert space with scalar product \(\langle \cdot, \cdot \rangle\). Let \(a \in H\) be a fixed element of \(H\). Prove that \(T(x) = \langle x, a \rangle\) is a continuous linear functional on \(H\) and find its norm. Is every continuous linear functional on \(H\) of this form?

(3) Consider the operator \(A\) defined on \(L^2[0, 1]\) by \(Af(x) = \int_0^x f(t) \, dt\).
   (a) Show that \(Af \in C[0, 1]\) whenever \(f \in L^2[0, 1]\).
   (b) Prove that \(A\) is a bounded linear operator on \(L^2[0, 1]\) (where \(L^2[0, 1]\) is equipped with the usual norm \(\|f\|_2^2 = \int_0^1 |f(x)|^2 \, dx\)). (Hint: You need not prove that \(A\) is linear.)
   (c) Find \(A^*\).

(4) Suppose that \(\{x_n\}_{n=1}^\infty\) is an orthonormal system in a real Hilbert space \(H\), \(x\) a vector in \(H\) and \(\{a_n\}_{n=1}^N\) a finite sequence of real numbers.
   (a) Show that
   \[
   \left\| x - \sum_{n=1}^N a_n x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2 + \sum_{n=1}^N |a_n - \langle x, x_n \rangle|^2.
   \]
   (b) Prove the following two identities as immediate consequences of the identity in part (a).
   (i) \(\left\| \sum_{n=1}^N a_n x_n \right\|^2 = \sum_{n=1}^N |a_n|^2\)
   (ii) \(\left\| x - \sum_{n=1}^N \langle x, x_n \rangle x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^N |\langle x, x_n \rangle|^2\).
   (c) Explain why the result in part (a) is sometimes called the \textit{Best Approximation Lemma}.