Linear Analysis Preliminary Exam

This exam consists of 4 questions.

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(1) Prove that if a metric space (M, d) is compact, and $f : M \to M$ is continuous with the property that $d(f(x), f(y)) < d(x, y)$ for all distinct $x, y \in M$, then f has a unique fixed point. Then give an example of a complete metric space (K, d^*) and a continuous function $g : K \to K$ so that $d^*(g(a), g(b)) < d^*(a, b)$ for all $a, b \in K$, but g has no fixed points.

(2) For $f \in C^1[0,1]$ define $\|f\|_1 = |f(0)| + \|f'\|_\infty$.

(a) Show that $\|\cdot\|_1$ defines a norm on $C^1[0,1]$.

(b) Is the norm $\|\cdot\|_1$ equivalent to the norm $\|\cdot\|_2$ defined as $\|f\|_2 = \|f\|_\infty + \|f'\|_\infty$? Justify your answer.

(3) Let X be a normed linear space. Give the definition of a linear functional on X and prove that a linear functional f is continuous if and only if it is bounded, i.e., there exists a nonnegative real number C such that $|f(x)| \leq C\|x\|$ for all $x \in X$.

(4) For normed linear spaces X and Y let $\mathcal{L}(X, Y)$ denote the space of bounded linear transformations from X to Y. For normed linear spaces X_1, X_2 and X_3 let $A \in \mathcal{L}(X_2, X_3)$ and $B \in \mathcal{L}(X_1, X_2)$.

(a) Prove that $AB \in \mathcal{L}(X_1, X_3)$ with $\|AB\| \leq \|A\|\|B\|$

(b) Give an example of a pair of bijective (that is, one-to-one and onto) operators A and B such that $\|AB\| = \|A\|\|B\|$.

(c) Give an example of a pair of bijective operators A and B such that $\|AB\| < \|A\|\|B\|$.
