Linear Analysis Preliminary Exam

This exam consists of 4 questions.

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(1) Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $0 < \lambda < 1$ is some constant.

(a) Prove that for any fixed $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to a fixed point c of T, i.e. a point c such that $T(c) = c$.

(b) Prove that T has only one fixed point.

(2) Let $X = C[-1, 1]$ be the Banach space of real-valued continuous functions on $[-1, 1]$ with the sup norm, i. e. $\|f\| = \max\{|f(x)| : -1 \leq x \leq 1\}, f \in X$. Define a linear functional $T : X \to \mathbb{R}$ by

$$Tf = \int_{-1}^{1} x^3 f(x)\, dx.$$

Prove that T is bounded and find the norm of T.

(3) Let P be a nonzero bounded linear operator on a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. Suppose that P is (a) self–adjoint and (b) satisfies $P^2 = P$.

(a) Prove that $\|Px\| \leq \|x\|$ for all $x \in H$ and that $\|P\| = 1$.

(b) Prove that each $x \in H$ can be written uniquely as $x = x_0 + x_1$ where $x_0 \in \mathit{Range}(P)$ and $x_1 \in \mathit{Range}(P)^{\perp}$. (In other words, that P is an orthogonal projector onto its range.)

(4) Let H be a real Hilbert space and $\{x_k : k = 1, 2, \ldots\}$ an orthonormal basis of H. Let n be a positive integer and let $c_1, \cdots, c_n$ be arbitrary real numbers. Let

$$y = \sum_{k=1}^{n} \langle x, x_k \rangle x_k \quad \text{and} \quad z = \sum_{k=1}^{n} c_k x_k.$$

Prove an identity relating $\|x - y\|^2$ and $\|x - z\|^2$, and conclude that $\|x - y\|^2 \leq \|x - z\|^2$.
