Linear Analysis Preliminary Exam

This exam consists of 5 questions.

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(1) Let $\mathbb{R}$ denote the set of real numbers, and for $x, y \in \mathbb{R}$, $d(x,y) = |x-y|$. Then we know that $(\mathbb{R}, d)$ is a complete metric space with the "usual" metric. Show that completeness is not preserved by homeomorphism, by finding a non-complete metric space (M, d^*) homeomorphic to $(\mathbb{R}, d)$ and an onto homeomorphism, $h : \mathbb{R} \to M$.

(2) (a) Let $C[0,1]$ denote the space of all continuous real-valued functions on $[0,1]$ equipped with the maximum norm $\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$. Show that $(C[0,1], \|\cdot\|_\infty)$ is not a Hilbert space, i.e., show that it is impossible to define a scalar product $(\cdot,\cdot)$ on $C[0,1]$ such that $\|f\|_\infty = (f,f)^{1/2}$ for all $f \in C[0,1]$.

(b) Define a scalar product on $C[0,1]$ (equipped with the usual addition and scalar multiplication of functions) and show that with this scalar product, $C[0,1]$ becomes a Euclidean space. Then show that the so-defined space is not complete.

(3) Let M be a finite-dimensional subspace of a normed linear space X. Show that there is a closed subspace $N \subset X$ with $X = M \oplus N$. (Hint: Given a basis $x_1, \ldots, x_n$ of M, find $f_1, \ldots, f_n \in X^*$ with $f_i(x_j) = \delta_{i,j}$.)

(4) Let $X = C[1,4]$ be the space of real–valued continuous functions on $[1,4]$ equipped with the maximum norm, $\|f\|_\infty = \max\{|f(x)| : x \in [1,4]\}$. For $f \in X$, define

$$T(f) = \int_1^2 f(x)\,dx - \int_3^4 f(x)\,dx$$

and $S(f) = f(2)$. Determine whether S and T are continuous. If a functional is continuous, find its norm; if not explain why not.

(5) Let $K(x,y)$ be a fixed function of two variables, continuous on the square $[0,1] \times [0,1]$, and let $A \in \mathcal{L}(L^2[0,1], L^2[0,1])$ be the operator defined by

$$Af(x) = \int_0^1 K(x,y)\,f(y)\,dy.$$

Prove that if $\{f_n\}_{n=1}^\infty$ satisfies $\|f_n\|_2 \le 1$ for all $n = 1, 2, \ldots$, then the set $\{Af_n\}_{n=1}^\infty$ is *equicontinuous*. This means that given $\epsilon > 0$, there is a $\delta > 0$ such that if $x, y \in [0,1]$ are such that $|x-y| < \delta$, then $|Af_n(x) - Af_n(y)| < \epsilon$ *for all* $n = 1, 2, \ldots$.
