Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2015

This exam consists of 5 questions.

---

1. Let $D_{15} = \{a^i b^j : a^{15} = e, \quad b^2 = e, \quad ba = a^{-1}b\}$ be the dihedral group of degree 15. How many elements does D_{15} have, and what are the orders of the elements of D_{15}? Justify your answer.

2. Show that any group of order 77 has a proper normal subgroup.

3. Recall that a *Euclidean ring* is an integral domain R equipped with a function $\phi : R \setminus \{0\} \to \mathbb{N}$ such that for all $a, b \in R$ with $a \neq 0$ there are $q, r \in R$ such that $b = qa + r$ with $r = 0$ or $0 < \phi(r) < \phi(a)$.

   (a) Show that every Euclidean ring is a PID (principal ideal domain).

   (b) Let $R = \mathbb{Z}[\gamma]$, where
   $$\gamma = \frac{1 + \sqrt{-3}}{2} \in \mathbb{C}.$$
   Using $\phi(a + b\gamma) := |a + b\gamma|^2 = a^2 + ab + b^2$ for integers a and b, show that R is a Euclidean subring of $\mathbb{C}$ and thereby a PID. Is R is a UFD (unique factorization domain)? Justify your answer.

4. Let k be a field and $R = k[X, Y]$ the polynomial ring over k in two variables X and Y. Without using that R is a UFD, prove explicitly that R is not a PID (*Hint:* You may want to consider the total degree of a polynomial in two variables.)

5. For an algebraic $\alpha \in \mathbb{C}$ let $R = \mathbb{Q}[\alpha]$ be the smallest subring of $\mathbb{C}$ that contains the rationals $\mathbb{Q}$ and α. Prove that R is a field, that is $R = \mathbb{Q}(\alpha)$. For $\alpha = \sqrt{2} + \sqrt{3}$ determine $[R : \mathbb{Q}]$, the dimension of R as a $\mathbb{Q}$-vector space.
