Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2014

This exam consists of 5 questions.

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1. Let $G = \langle x \rangle$ be a cyclic group of order $n \in \mathbb{N}$. For and arbitrary integer $k \in \mathbb{Z}$, what is the order of the element x^k?

2. Show that any group of order 45 is abelian, and that there are exactly two such groups that are non-isomorphic.

3. let R be a set given by
$$R = \left\{ \begin{pmatrix} n & m \\ m & 2m+n \end{pmatrix} : n, m \in \mathbb{Z}_3 \right\}.$$
Prove that R is a finite ring, with the usual matrix addition and multiplication. Is R commutative? How many elements does R have? Determine the units of R.

4. For an abelian group G, show that the set $\mathrm{End}(G)$ of group endomorphism $G \to G$ forms a ring with componentwise addition and map composition. Let p and q be distinct primes. Describe the ring $\mathrm{End}(\mathbb{Z}_p \times \mathbb{Z}_q)$ in terms of familiar rings.

5. Let S be a PID and $a, b \in S$ be relatively prime elements. Show that $S/(ab) \cong S/(a) \oplus S/(b)$ as rings. Let $\mathbb{R}$ denote the field of real numbers. For $n \in \mathbb{N}$ let $f(X) = \prod_{k=0}^{n}(X^2 + X + k) \in \mathbb{R}[X]$. Show that $\mathbb{R}[X]/(f(X))$ is a direct product of fields. How many fields are there in this product, and how many occurrences are there of each field?
