Department of Mathematical Sciences George Mason University, January 2011

Algebra Preliminary Exam

This exam consists of 5 questions.

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(1) Let K be a field. Define $K^* = (K \setminus \{0\}, \cdot)$, the nonzero elements of K.

(a) Show that K^* under the operation of multiplication is an abelian group.

(b) Show that none of the groups $\mathbb{Q}^*$, $\mathbb{R}^*$, and $\mathbb{C}^*$ are isomorphic.

(2) For a group G let $N_G = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$ be the commutator subgroup of G.

(a) Show that N_G is normal in G.

(b) Show that G/N_G is abelian.

(c) Give an example of an infinite group G such that G/N_G is finite.

(3) Let $R = \mathbb{Z}[\sqrt{-2}] = \{m + n\sqrt{-2} : m, n \in \mathbb{Z}\}$.

(a) Show that R is a principal ideal domain.
(*Hint:* Show first that R is a Euclidean ring with $\phi(a + \sqrt{2}b) = a^2 + 2b^2$.)

(b) What are the units of R?

(4) For $n \in \mathbb{N}$ let $f_n(X) \in \mathbb{Z}[X]$ be given by

$$f_n(X) = \frac{X^n - 1}{X - 1} = X^{n-1} + X^{n-2} + \cdots + X + 1.$$

Show that $f_n(X)$ is irreducible over $\mathbb{Z}$ if and only if n is a prime number.

(5) Let F be the field $F = \mathbb{Q}(i, \sqrt{3})$.

(a) Find $[F : \mathbb{Q}]$, the degree of the field extension F over $\mathbb{Q}$.

(b) Find a basis for F over $\mathbb{Q}$.
