Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam (January 2010)

This exam consists of 5 questions.

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(1) Determine all the automorphisms of the additive group $\mathbb{Z}_{100} = \mathbb{Z}/100\mathbb{Z}$. How many are they?

(2) Show that any group of order pqr where p, q and r are distinct primes has a nontrivial proper normal subgroup. (*Hint:* Consider the number of its p-, q- and r-Sylow subgroups.)

(3) Let $R = \mathbb{Z}[\sqrt{-5}] = \{m + n\sqrt{-5} : m, n \in \mathbb{Z}\}$ and $I = \{m + n\sqrt{-5} : m - n \text{ is even }\}$. Show that I is a maximal ideal. Identify to which field R/I is isomorphic.

(4) Let R be a commutative ring. Show that every maximal ideal is prime. Give an example of a commutative ring and a nonzero prime ideal that is not maximal.

(5) Let R be an integral domain containing the field K as a subring. If R is finitely dimensional over K as a vector space, show that R is indeed a field.
