Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam
August, 2010

This exam consists of 5 questions.

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(1) Let p be a prime number and $\mathbb{Z}_p$ the additive group of integers modulo p.

(a) Show that $|Aut(\mathbb{Z}_p)| = p - 1$.

(b) Use this to prove *Fermat's Little Theorem:* $a^{p-1} \equiv 1$ modulo p for each nonzero $a \in \mathbb{Z}_p$.

(2) Let G be a finite group and N a normal p-Sylow subgroup of G. If $f : G \to G$ is a group endomorphism, show that $f(N)$ is a subgroup of N. (*Hint*: First show that $f(N)$ is contained in a p-Sylow subgroup.)

(3) Let R be the ring of all continuous functions $f : [a, b] \to \mathbb{R}$. For $c \in [a, b]$ let $I_c = \{f \in R : f(c) = 0\}$.

(a) Show that I_c is a maximal ideal of R for each c.

(b) Must every maximal ideal of R be of the form I_c for some c?

(4) Let R be a commutative ring. Let

$$\mathcal{N}_R = \{x \in R : x^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

(a) Show that $\mathcal{N}_R$ is an ideal of R.

(b) Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. Show that the ring $R/\mathcal{N}_R$ has no nilpotent elements.

(5) Find the degree of the following extension fields over the rationals $\mathbb{Q}$. Justify your answer.

(a) The field $F_1 = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(b) The smallest field F_2 containing all roots of the polynomial $x^2 - 6$.
