Algebra Preliminary Exam

This exam consists of 5 questions.

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(1) Let p be a prime and $k \in \mathbb{N}$. Determine all the distinct isomorphism classes of abelian groups of order p^k. Prove explicitly that no two of them are isomorphic.

(2) Let G be a group with center $C(G)$. Show that if $G/C(G)$ is cyclic, then G is abelian.

(3) Let R be a ring given by

$$R = \left\{ \begin{pmatrix} n & -m \\ m & n \end{pmatrix} : n, m \in \mathbb{Z}_7 \right\}$$

where the multiplication is given by the usual matrix multiplication. Prove that R is a finite field.

(4) Let R be an integral domain and $R[X]$ be the ring of polynomials over R. Determine all the automorphisms $R[X] \to R[X]$ that fix R.

(5) Show that we have the following isomorphism of rings.

$$\mathbb{R}[X]/((X-1)(X^2+1)) \cong \mathbb{R} \times \mathbb{C}.$$
