Algebra Preliminary Exam

This exam consists of 5 questions.

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(1) Let G be a group of order 6. Show that either G is cyclic or isomorphic to S_3, the symmetric group on $\{1, 2, 3\}$.

(2) Let H and K be subgroups of a given group G. Prove that $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup of G if either H or K is normal in G. For G finite, determine how many elements HK has. Prove your answer.

(3) let R be a ring given by

$$R = \left\{ \begin{pmatrix} n & -m \\ m & n \end{pmatrix} : n, m \in \mathbb{Z}_5 \right\},$$

with the usual matrix addition and multiplication. Prove that R is a finite commutative ring. How many elements does R have? Is R a field? Be sure to prove your answer.

(4) Let K be a field and let $R = K[[X]]$ be the ring of all formal power series over K. Recall that this consists of all formal infinite sums

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

with $a_i \in K$. Multiplication and addition are defined by extending the usual polynomial multiplication and addition.

a. Show that R is a PID.

b. Prove that $1 - x$ is invertible by explicitly finding its inverse.

c. Prove that the units of R are the formal power series with nonzero constant term.

(5) Let G be a finite group of order 500. State Sylow's subgroup theorem, and use it to prove that G has an element of order 5.
