Algebra Preliminary Exam

This exam consists of 5 questions.

---

(1) Prove that a group of order 5 or less is abelian.

(2) Let G be a nonabelian group of order 36. Show that G must have more than one Sylow 2-subgroup or more than one Sylow 3-subgroup.

(3) Let R be a commutative ring.

(a) Show that every maximal ideal is prime.

(b) If further R is a PID, show that every nonzero prime ideal is maximal.

(4) Let $R = \mathbb{Z}[\sqrt{3}] = \{m + n\sqrt{3} : m, n \in \mathbb{Z}\}$. Show that R is a subring of $\mathbb{R}$. Show that $m + n\sqrt{3}$ is a unit if, and only if, m^2 equals either $3n^2 - 1$ or $3n^2 + 1$.

(5) Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers and let $N \subseteq \mathbb{C}^*$ be the set of complex numbers of absolute value 1 (that is, $a + bi \in N$ if, and only if, $a^2 + b^2 = 1$.) Prove that N is a subgroup of $\mathbb{C}^*$ and that $\mathbb{C}^*/N$ is isomorphic to $(\mathbb{R}^+, \cdot)$, the multiplicative group of the positive real numbers.
