Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2017

This exam consists of 5 questions, all of equal weight.

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1. Let p and q be two distinct odd primes. Let $D_{pq} = \{a^i b^j : a^{pq} = e, \quad b^2 = e, \quad ba = a^{-1}b\}$ be the dihedral group of degree pq. What are the possible orders of an element of D_{pq}? Justify your answer.

2. Let H and K be finite normal subgroups of G of relative prime orders such that $G = HK$. Show that $G \cong H \times K$,

3. Let $R = \mathbb{Z}[\sqrt{2}]$ be the smallest subring of $\mathbb{R}$ containing $\mathbb{Z}$ and $\sqrt{2}$.
   (a) Show explicitly that $R = \{m + n\sqrt{2} : n, m \in \mathbb{Z}\}$ as a set.
   (b) Show that the *norm* $N(m + n\sqrt{2}) = m^2 - 2n^2$ is well defined and respects multiplication; $N(xy) = N(x)N(y)$ for any $x, y \in R$.
   (c) Show that an element $x \in R$ is a unit if and only if $N(x) = \pm 1$. (*Hint:* factor $m^2 - 2n^2$ over the reals.)

4. Let $R = \mathbb{C}[X, Y]$, the polynomial ring over the complex numbers in two variables. Show that $I = (Y - X^2)$ is a prime ideal of R but not maximal. Find two maximal ideals of R that contain I, and show that the ideals you find are indeed maximal and contain I.

5. Let $F = \mathbb{Q}(\sqrt{2}, i)$ be the smallest subfield of $\mathbb{C}$ that contains $\mathbb{Q}$, $\sqrt{2}$ and the complex number i. Compute $[F : \mathbb{Q}]$, the degree of the field extension F over $\mathbb{Q}$. Justify your answer.
