Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2016

This exam consists of 5 questions.

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1. Show that a subgroup of a cyclic group is itself cyclic. Determine (with proof) whether the following additive groups are cyclic: $\mathbb{Z}_n \times \mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Q}$.

2. For a prime p, show that any group of order p^2 is either cyclic or isomorphic to the additive group $\mathbb{Z}_p \times \mathbb{Z}_p$, and hence abelian.

3. Let R be the smallest subring of $\mathbb{C}$ that contains the integers and $\sqrt{-5}$. Show that $R = \mathbb{Z}[\sqrt{-5}] = \{m + n\sqrt{-5} : m, n \in \mathbb{Z}\}$. Let $N : R \to \mathbb{Z}$ be the *norm* given by $N(m + n\sqrt{-5}) := m^2 + 5n^2$. Show that $N(xy) = N(x)N(y)$ for all $x, y \in R$, and that the only units in R are ± 1.

4. In the ring $R = \mathbb{Z}[\sqrt{-5}]$ from above, factor $6 \in R$ in two ways into a product of irreducible elements, thereby proving that R is not a UFD.

5. Assuming the field $\mathbb{C}$ of the complex numbers is algebraically closed, prove that every irreducible polynomial $p(X) \in \mathbb{R}[X]$ has degree of either one or two.
