Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2014

This exam consists of 5 questions.

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1. Let H and K be subgroups of a given group G. Prove that $HK = \{hk : h \in H,\ k \in K\}$ is a subgroup of G if either H or K is normal in G. For G finite, determine the order of HK. Prove your answer.

2. Let G be a nonabelian group of order 36. Show that G must have more than one Sylow 2-subgroup or more than one Sylow 3-subgroup.

3. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the set of polynomials over the rationals whose constant term is an integer. Then R is an integral domain.

   (a) Show that the only units of R are ± 1.

   (b) Show that x can be factored into ab, where neither a nor b is a unit, in infinitely many ways. (Hence R is not a UFD.)

4. Let R be a commutative ring. Let

$$\mathcal{N}_R = \{x \in R : x^n = 0 \text{ for some } n \in \mathbb{N}\}.$$

   (a) Show that $\mathcal{N}_R$ is an ideal of R.

   (b) Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$. Show that the ring $R/\mathcal{N}_R$ has no nonzero nilpotent elements.

5. Let R be a integral domain containing the field K as a subring. If R is finite dimensional over K as a vector space, show that R is indeed a field. (Hint: For any $a \in R$ there is a linear relation over K on the set $\{1, a, a^2, \dots\}$ - why?)
