Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2012

This exam consists of 5 questions.

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1. Show that all groups of order 5 or less are abelian (you may use Lagrange's Theorem).

2. Let G be a finite group.

   (a) Suppose that G is the (internal) direct sum of subgroups H and K, such that the orders of H and K are relatively prime. Show that any subgroup of G is of the form $H' \times K'$, where H' is a subgroup of H and K' is a subgroup of K.

   (b) G is called *catenarian* if every maximal chain of subgroups has the same length. Let G be a group of order q^2p^2 where q and p are distinct primes. If both the Sylow p-subgroup and Sylow q-subgroup are normal, show that G is catenarian.

3. Let N be a normal subgroup of G of finite index $[G : N]$. Let H be a finite subgroup of G such that the order $|H|$ and the index $[G : N]$ are relatively prime integers. Show that H is a subgroup of N.

4. Let $R = \mathbb{Z} + x\mathbb{Q}[x] \subset \mathbb{Q}[x]$ be the set of polynomial in x with rational coefficients whose constant term is an integer. Then R is an integral domain whose only units are ± 1 (you do not need to show this).

   (a) Show that x can be factored into ab, where neither a nor b is a unit in infinitely many ways. (Hence R is not a UFD.)

   (b) Show that the only irreducibles in R are elements of the form $\pm p$ where p is a prime number and those polynomials that are irreducible in $\mathbb{Q}[x]$ and have constant term ± 1.

5. Let R be the ring given by

$$R = \left\{ \begin{pmatrix} n & -m \\ m & n \end{pmatrix} : n, m \in \mathbb{Z}_7 \right\}$$

   where the multiplication is given by the usual matrix multiplication. Prove that R is a field. Determine the order of R.
