Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, January 2018

This exam consists of 5 questions, all of equal weight.

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(1) For $n \geq 3$ the *dihedral group* D_n of degree n is given by $D_n = \langle a^i b^j : a^n = e, \ b^2 = e, \ ba = a^{-1}b \rangle$. Determine with proof those n for which D_n is isomorphic to the symmetric group S_n, that is $D_n \cong S_n$.

(2) Fix a prime p and a positive integer k. For groups $G_1, \ldots, G_k$ the Cartesian product $G_1 \times \cdots \times G_k$ is naturally a group by applying the operations component-wise.

(a) If all the groups $G_1, \ldots, G_k$ are finite and H_i is a p-Sylow subgroup of G_i for each i, show that $H_1 \times \cdots \times H_k$ is p-Sylow subgroup of $G_1 \times \cdots \times G_k$.

(b) Use a Sylow Theorem to show that any p-Sylow subgroup of the Cartesian product $G_1 \times \cdots \times G_k$ has the form from above: $H_1 \times \cdots \times H_k$ where H_i is a p-Sylow subgroup of G_i for each i.

(3) Let R be a principal ideal domain and x be a nonzero element of R. Let $I = \langle x \rangle$ be the ideal in R generated by x. Then I is a maximal ideal in R if and only if x is irreducible in R.

(4) Let $\mathbb{Q}[\sqrt[3]{3}] = \{a + b\sqrt[3]{3} + c(\sqrt[3]{3})^2 \ : \ a, b, c \in \mathbb{Q}\}$ be a ring under the multiplication and addition operations inherited from the real numbers. Let $f : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt[3]{3}]$ be given by the extension of $1 \mapsto 1$, $x \mapsto \sqrt[3]{3}$ to a ring homomorphism on all of $\mathbb{Q}[x]$.

(a) What is the kernel of this map? What is the image?

(b) What can you conclude, based on the First Isomorphism Theorem for Rings?

(c) The ring $\mathbb{Q}[\sqrt[3]{3}]$ forms a field. What conclusion can be made about the ideal you found in part (a)?

(5) Let $n \in \mathbb{Z}$ and let $f(x) = x^3 - nx^2 - (n+3)x - 1$.

(a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) Show that if r is a root of $f(x)$ then $-1/(1+r)$ is also a root of $f(x)$.

(c) Compute the smallest field extension of $\mathbb{Q}$ where $f(x)$ factors completely into linear factors.
