Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2016

This exam consists of 4 questions.

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1. For a positive integer $n \geq 2$ let $U(n) = \{\overline{a} \in \mathbb{Z}/n\mathbb{Z} : \gcd(a, n) = 1\}$.

   (a) Show that $U(n)$ is a group with respect to multiplication in $\mathbb{Z}/n\mathbb{Z}$ inherited from $\mathbb{Z}$.

   (b) If n is not divisible by 2 or 5, then use Lagrange's theorem on $U(n)$ to show that the rational number $1/n$ can be written as a repeating decimal:

   $$0.\overline{d_1 d_2 \cdots d_k} = 0.d_1 d_2 \cdots d_k d_1 d_2 \cdots d_k d_1 d_2 \cdots d_k \cdots ,$$

   where the d_i are decimal digits. (*Hint:* First show directly that $1/(10^\ell - 1)$ is the repeating decimal corresponding to the segment of digits consisting of $\ell - 1$ zeros and followed by a one.)

   (c) Use the above to conclude that *any* positive rational number may be written as either a terminating decimal or an *eventually* repeating decimal; that is, it may be written as $a_1 \cdots a_k.b_1 \cdots b_\ell \overline{c_1 \cdots c_m}$, where the a_i, b_i and c_i are decimal digits.

2. How many elements of order 7 are there in a non-abelian group of order 168? (Note! There are two possible answers.)

3. Consider $R = \mathbb{C}[X, Y]$, the polynomial ring over the complex numbers in two variables. Show that $I = (XY - 1) \subseteq R$ is a prime ideal, but not maximal. Find two distinct maximal ideals of R that contain I.

4. Prove that any finite field has cardinality p^n for some prime p and natural number n.
