Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2015

This exam consists of 5 questions.

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1. Prove that every element of the symmetric group S_n of degree $n \geq 2$ is a product of 2-cycles of the form $(1, k)$.

2. Let G be a group of order $|G| = pq$ where $p > q$ are distinct primes and q does not divide $p-1$. Show that G is cyclic.

3. For integers $n \geq 2$ determine all the prime ideals of the factor ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Is every prime ideal of $\mathbb{Z}_n$ also maximal? Justify your answer.

4. Let R be a commutative ring and $I_1, \dots, I_k$ ideals of R. Let $\phi : R \to R/I_1 \times \cdots \times R/I_k$ be the natural diagonal ring homomorphism given by $\phi(x) = (x + I_1, \dots, x + I_k)$.

   (a) Show that the kernel is given by $\ker(\phi) = I_1 \cap \cdots \cap I_k$.

   (b) Show that ϕ is surjective if and only if $I_i + I_j = R$ for every $i \neq j$, and in this case prove the *Chinese Reminder Theorem* for rings: $R/(I_1 \cap \cdots \cap I_k) \cong R/I_1 \times \cdots \times R/I_k$.

5. Recall that for a field k and elements $a_1, \dots, a_n$ in a field K that contains k, then the *field extension* $k(a_1, \dots, a_n)$ *of* k denotes the smallest subfield of K that contains all of k and the elements $a_1, \dots, a_n$.

   For $\alpha, \beta \in \mathbb{Q} \setminus \{0\}$, is it always the case for the the following field extensions of $\mathbb{Q}$ that $\mathbb{Q}(\sqrt{\alpha}, \sqrt{\beta}) = \mathbb{Q}(\sqrt{\alpha} + \sqrt{\beta})$? Justify your answer.
