Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2018

This exam consists of 5 questions.

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1. Let r, n be integers with $1 < r \leq n$ and let S_n denote the symmetric group on n letters. Let $i_1, \ldots, i_r$ be distinct integers with $1 \leq i_j \leq n$ for all j. Let $\tau \in S_n$ and consider the r-cycle $\sigma := (i_1 i_2 \cdots i_r) \in S_n$. Prove that $\tau\sigma\tau^{-1}$ is the r-cycle $(\tau(i_1)\tau(i_2)\cdots\tau(i_r))$.

2. Let G be an abelian group and let H be the subset of G consisting of those elements that have finite order.

   (a) Prove that H is a subgroup of G.

   (b) Prove that any nonidentity element of the quotient group G/H has infinite order as an element of G/H.

   (c) Give an example to show that if one removes the assumption of abelianness from G, then H is not necessarily a group.

3. Let R be a unique factorization domain. Let $a, b \in R$, and let d be a greatest common divisor of a and b. Let $I := (a, b)$, the ideal generated by these elements, and let J be the intersection of all principal ideals that contain I.

   (a) Prove that $J = (d)$.

   (b) Give an example to show that I need not equal J.

4. Let $B = \mathbb{R}[t]$ and $A = \{f \in B \mid f(1) = f(-1)\}$.

   (a) Prove that A is a subring of B.

   (b) Prove that any polynomial in A that is of degree 2 or 3 is irreducible.

   (c) On the other hand, prove that the elements $\alpha = t^2$, $\beta = t^3 - t$ are not prime as elements of A. [Hint: Find two nonequivalent factorizations of the element $t^6 - 2t^4 + t^2$ of A.]

5. Let K be a field, let t be an indeterminate over K. Let $h = h(t) \in K[t] \setminus K$. Prove that h is transcendental over K.
