Algebra Preliminary Exam, August 2013

This exam consists of 5 questions.

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1. Let $\varepsilon : (\mathbb{R}, +) \to (\mathbb{C}^*, \cdot)$ be defined by $\varepsilon(\alpha) := \cos(2\pi\alpha) + i\sin(2\pi\alpha) = e^{2\pi\alpha i}$ (with all angles given in radians and $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$). Show that ε is a group homomorphism. What is its image? What is its kernel? What two groups can you conclude are isomorphic to each other?

2. Let G be a non-abelian group of order 55.
   (a) How many subgroups are there of each possible order?
   (b) How many elements are there of each possible order?

3. Let R be a commutative ring with 1, and let X be the set of prime ideals of R. For any ideal I of R, let $V(I) := \{P \in X \mid I \subseteq P\}$.
   (a) Let $\{I_\alpha\}_{\alpha \in S}$ be a family of ideals of R. Show that $V(\sum_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$
   (b) Let P be a prime ideal. Suppose $IJ \subseteq P$, where I, J are ideals. Show that $I \subseteq P$ or $J \subseteq P$.
   (c) Show that $V(IJ) = V(I) \cup V(J)$ for any pair I, J of ideals of R.

4. (a) Let $f(x), g(x) \in \mathbb{R}[x]$ be relatively prime and of positive degree (where for the purposes of this problem, $\deg 0 = -\infty$), and let $m(x)$ be a polynomial with $\deg m < \deg f + \deg g$. Show that there is a pair of polynomials $n(x), p(x) \in \mathbb{R}[x]$ with $\deg n < \deg f$, $\deg p < \deg g$, and
   $$\frac{m(x)}{f(x)g(x)} = \frac{n(x)}{f(x)} + \frac{p(x)}{g(x)}.$$
   (b) Use part (a) to show that given any k-tuple $a_1, \ldots, a_k$ of distinct real numbers, there is a k-tuple $A_1, \ldots, A_k$ of (not necessarily distinct) real numbers such that in $\mathbb{R}[x]$,
   $$\frac{1}{(x-a_1)(x-a_2)\cdots(x-a_k)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k}.$$

5. Let F be a field and $h(x) \in F[x]$.
   (a) Show that if $h(x) \neq 0$, the number of roots of h is bounded above by $\deg h$.
   (b) If F is infinite and $h(a) = 0$ for all $a \in F$, prove that $h(x)$ is the zero polynomial.
   (c) Give an example of a finite field F and $0 \neq h(x) \in F[x]$ such that $h(a) = 0$ for all $a \in F$.
