Department of Mathematical Sciences George Mason University

# Algebra Preliminary Exam, August 2011

This exam consists of 5 questions.

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1. Prove that all subgroups of order 12 have a proper nontrivial normal subgroup.
2. Let $G$ be a group of order $n$. If $p$ is the smallest prime divisor of $n$, prove that any subgroup of $G$ of index $p$ is normal.
3. Prove that $S_n$ is generated by $\{(i\ i+1) : 1 \leq i \leq n-1\}$. (Consider conjugates)
4. Let $I$ be an ideal of a commutative ring $R$ and define

$$\text{rad } I = \{r \in R \mid r^n \in I \text{ for some } n > 0\}$$

called the nilradical of $I$.

   (a) Prove that rad $I$ is an ideal of $R$ containing $I$.

   (b) Prove that $(\text{rad } I)/I = \mathfrak{N}(R/I)$ where $\mathfrak{N}(S)$ denotes the set of nilpotent elements of a commutative ring $S$. (An element $x$ of a ring is called *nilpotent* if $x^n = 0$ for some $n > 0$.)

5. Let $F$ be a field. Let $f(x)$ and $g(x)$ be relatively prime elements of $F[X]$. Show that $f(x)$ and $g(x)$ do not have a common root in $F$.

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