Algebra Preliminary Exam

This exam consists of 5 questions.

(1) Prove that any finite group $G$ is a subgroup of the symmetric group $S_n$ for some $n$.

(2) Let $G_1$ and $G_2$ be two finite groups. If $H_i$ is a $p$-Sylow subgroup of $G_i$ for $i = 1, 2$, show that $H_1 \times H_2$ is a $p$-Sylow subgroup of $G_1 \times G_2$. Use the first part and the Sylow Theorems to prove that every $p$-Sylow subgroup of $G_1 \times G_2$ has the form $H_1 \times H_2$ where $H_i$ is a $p$-Sylow subgroup of $G_i$ for $i = 1, 2$.

(3) Let $R$ be a ring. Suppose that $e \in R$ is a central idempotent, that is $ea = ae$ for all $a \in R$ and that $e^2 = e$.
   1. Prove that $e' = 1 - e$ is also a central idempotent of $R$.
   2. Show that both $R_1 = Re$ and $R_2 = Re'$ are rings with identities $e$ and $e'$ respectively.
   3. Show that $R \cong R_1 \times R_2$ as rings.

(4) Let $R$ be a commutative ring. If $R$ is a field, prove that $R[X]$ has infinitely many irreducible elements. For what general class of rings does the proof hold and why?

(5) Let $R = \mathbb{Q}[\sqrt[3]{3}] = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} : a, b, c \in \mathbb{Q}\}$. Show that for any $r \in R$ there is a polynomial $f(X) \in \mathbb{Z}[X]$ such that $f(r) = 0$. 
