Algebra Preliminary Exam

This exam consists of 5 questions.

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(1) Prove that any finite group G is a subgroup of the symmetric group S_n for some n.

(2) Let G_1 and G_2 be two finite groups. If H_i is a p-Sylow subgroup of G_i for $i = 1, 2$, show that $H_1 \times H_2$ is a p-Sylow subgroup of $G_1 \times G_2$. Use the first part and the Sylow Theorems to prove that every p-Sylow subgroup of $G_1 \times G_2$ has the form $H_1 \times H_2$ where H_i is a p-Sylow subgroup of G_i for $i = 1, 2$.

(3) Let R be a ring. Suppose that $e \in R$ is a central idempotent, that is $ea = ae$ for all $a \in R$ and that $e^2 = e$.

1. Prove that $e' = 1 - e$ is also a central idempotent of R.
2. Show that both $R_1 = Re$ and $R_2 = Re'$ are rings with identities e and e' respectively.
3. Show that $R \cong R_1 \times R_2$ as rings.

(4) Let R be a commutative ring. If R is a field, prove that $R[X]$ has infinitely many irreducible elements. For what general class of rings does the proof hold and why?

(5) Let $R = \mathbb{Q}[\sqrt[3]{3}] = \{a + b\sqrt[3]{3} + c\sqrt[3]{9} : a, b, c \in \mathbb{Q}\}$. Show that for any $r \in R$ there is a polynomial $f(X) \in \mathbb{Z}[X]$ such that $f(r) = 0$.
