Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2019

This exam consists of 5 questions, all of equal weight.

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1. Let $\phi : G \to H$ be a group homomorphism. Prove:
   (a) $\ker(\phi) \triangleleft G$,
   (b) $\phi(G) \leq H$,
   (c) $G/\ker(\phi) \cong \phi(G)$ as groups.
2. Show that any group of order 99 is abelian.
3. Let R be a commutative ring with identity, I an ideal of R, and $\mathrm{rad}(I) = \{r \in R : r^n \in I \text{ for some } n > 0\}$.
   (a) Prove that $\mathrm{rad}(I)$ is an ideal of R containing I.
   (b) Prove that the ideal $\mathrm{rad}(I)/I$ of R/I is equal to the set of nilpotent elements of R/I (elements $x \in R/I$ such that $x^n = 0$ for some $n \geq 0$).
4. Let R be an integral domain that is not a field. Let $S = R[x]$ be the ring of polynomials in x with coefficients in R. Let $r \in R$ be a nonzero element that is not a unit. Prove that the ideal (r, x) of S is *not* principal. Conclude that S is not a PID.
5. Let $R = \mathbb{Q}[\sqrt{2}+\sqrt{3}]$ be the smallest subring of $\mathbb{R}$ containing $\mathbb{Q}$ and $\sqrt{2}+\sqrt{3}$. Prove that R is a field, and determine $[R : \mathbb{Q}]$, the dimension of R as a $\mathbb{Q}$-vector space.