Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam, August 2017

This exam consists of 5 questions, all of equal weight.

---

1. Classify all groups up to order 11. You must fully justify your claims with theorems or proofs. You do not have to prove the theorems you use, but you must clearly state them.

2. Let G be a group, and let $p < q$ be distinct primes such that $q \not\equiv 1 \mod p$.

   (a) Suppose H is a subgroup of the center of G. Prove that, if G/H is cyclic, then G is abelian.

   (b) Suppose G has order pq. Prove that G is cyclic.

   (c) Let r be a distinct prime from p and q above, and assume that G is a non-abelian group of order pqr. Prove that the order of the center of G is $1, p$, or q.

3. Show that in a UFD an element is irreducible if and only if it is prime. Given an example of integral domain containing an irreducible element that is not prime, and identify the element.

4. Let $\mathbb{F}$ be a field. Prove that $\mathbb{F}[x]$ is a PID. Show that $\mathbb{Z}[x]$ is not a PID.

5. Let $\mathbb{F}_q$ be a field of order q and let $\mathrm{SL}(2, \mathbb{F}_q)$ be the group of 2×2 matrices over $\mathbb{F}_q$ with determinant 1, that is,

$$\mathrm{SL}(2, \mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{F}_q, \quad ad - bc = 1 \right\}.$$

   For any $g \in \mathrm{SL}(2, \mathbb{F}_q)$, recall the characteristic equation of g is $x^2 - \mathrm{tr}(g)x + 1 = 0$.

   (a) Compute the order of the group $\mathrm{SL}(2, \mathbb{F}_q)$.

   (b) Suppose that the characteristic equation for g has *distinct* solutions in $\mathbb{F}_q$. Prove that $|g|$ divides $q - 1$.

   (c) Suppose that there are no solutions for the characteristic equation in $\mathbb{F}_q$. Find a field extension over which there are solutions, and identify its degree over $\mathbb{F}_q$.

   (d) In the case that the characteristic equation for g does not have solutions in $\mathbb{F}_q$, show that $|g|$ divides $q + 1$.