Algebra Preliminary Exam (January 2009)

This exam consists of 5 questions.

(1) Let $G$ be a group, $S \subseteq G$ a subset and $C_G(S) = \{ g \in G : gs = sg \text{ for every } s \in S \}$ the centralizer of $S$ in $G$. Show that $C_G(S)$ is a subgroup of $G$. If $S_1 \subseteq S_2$, what is the relation between $C_G(S_1)$ and $C_G(S_2)$? What must $S$ satisfy for $C_G(S)$ to be normal in $G$?

(2) Let $G$ be a group with $|G| = p^n m$ where $p$ is a prime, gcd($p, m$) = 1 and $p > m - 1$. Show that $G$ has a unique $p$-Sylow subgroup $H$ and that $H < G$ is normal.

(3) Let $R$ be a commutative ring with a prime characteristic $p$ (so $p = p \cdot 1_R = 0_R$ in $R$) Show that the map $\phi : R \rightarrow R$ given by $\phi(x) = x^p$ is a ring homomorphism.

(4) Define what it means for an integral domain to be a Euclidean ring. Prove that any Euclidean ring $R$ is a PID.

(5) Let $K$ be a field and $M$ a vector space over $K$ such that $M$ contains a finite set $S$ such that $M = \text{Span}_K(S)$. Show that $M$ has a finite basis of cardinality at most $|S|$.