

Algebra Preliminary Exam (January 2009)

This exam consists of 5 questions.

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- (1) Let G be a group, $S \subseteq G$ a subset and $C_G(S) = \{g \in G : gs = sg \text{ for every } s \in S\}$ the *centralizer* of S in G . Show that $C_G(S)$ is a subgroup of G . If $S_1 \subseteq S_2$, what is the relation between $C_G(S_1)$ and $C_G(S_2)$? What must S satisfy for $C_G(S)$ to be normal in G ?
 - (2) Let G be a group with $|G| = p^n m$ where p is a prime, $\gcd(p, m) = 1$ and $p > m - 1$. Show that G has a unique p -Sylow subgroup H and that $H \triangleleft G$ is normal.
 - (3) Let R be a commutative ring with a prime characteristic p (so $p = p \cdot 1_R = 0_R$ in R) Show that the map $\phi : R \rightarrow R$ given by $\phi(x) = x^p$ is a ring homomorphism.
 - (4) Define what it means for an integral domain to be a Euclidean ring. Prove that any Euclidean ring R is a PID.
 - (5) Let K be a field and M a vector space over K such that M contains a finite set S such that $M = \text{Span}_K(S)$. Show that M has a finite basis of cardinality at most $|S|$.
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