Algebra Preliminary Exam

This exam consists of 5 questions.

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(1) Let G be a group and N a normal subgroup of G. Let $G/N = \{gN : g \in G\}$ be the set of cosets. Show that G/N is a group under the operation $gN \cdot g'N = gg'N$.

(2) Let G be an abelian group and $n \in \mathbb{N}$. Let $G_n = \{g \in G : g^n = e\}$ where $e \in G$ is the identity element of G and let $G^n = \{g^n : g \in G\}$. Show that $G/G_n \cong G^n$.

(3) Let p be a prime number.

(a) Show that the center of a finite p-group is nontrivial.

(b) Show that any group of order p^2 is abelian.

Prove any lemmas you use.

(4) Let $\phi : R \to R'$ be a ring homomorphism. Show that $\ker(\phi)$ is an ideal of R and that the image $\phi(R)$ is a subring of R'.

(5) Let α be a irrational real root of the polynomial $X^2 + bX + c$ where b and c are rational. Show that $\mathbb{Q}[\alpha]$, the smallest subring of $\mathbb{R}$ containing the rationals $\mathbb{Q}$ and α, is a field.
