Department of Mathematical Sciences George Mason University

Algebra Preliminary Exam (January 2009)

This exam consists of 5 questions.

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(1) Let G be a group, $S \subseteq G$ a subset and $C_G(S) = \{g \in G : gs = sg \text{ for every } s \in S\}$ the *centralizer* of S in G. Show that $C_G(S)$ is a subgroup of G. If $S_1 \subseteq S_2$, what is the relation between $C_G(S_1)$ and $C_G(S_2)$? What must S satisfy for $C_G(S)$ to be normal in G?

(2) Let G be a group with $|G| = p^n m$ where p is a prime, $\gcd(p, m) = 1$ and $p > m - 1$. Show that G has a unique p-Sylow subgroup H and that $H \lhd G$ is normal.

(3) Let R be a commutative ring with a prime characteristic p (so $p = p \cdot 1_R = 0_R$ in R) Show that the map $\phi : R \to R$ given by $\phi(x) = x^p$ is a ring homomorphism.

(4) Define what it means for an integral domain to be a Euclidean ring. Prove that any Euclidean ring R is a PID.

(5) Let K be a field and M a vector space over K such that M contains a finite set S such that $M = \mathrm{Span}_K(S)$. Show that M has a finite basis of cardinality at most $|S|$.
