Numerical Analysis Preliminary Examination questions
August 2012

Instructions: NO CALCULATORS are allowed. This examination contains **seven** problems, each worth 20 points. Do any **five** of the seven problems. Show your work. Clearly indicate which **five** are to be graded.

PLEASE GRADE PROBLEMS: 1 2 3 4 5 6 7

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1. Let a_i^* for $i = 1, 2, 3, 4$ be positive numbers on a computer. With a unit round-off error δ, $a_i^* = a_i(1 + \epsilon_i)$ with $|\epsilon_i| \leq \delta$, where a_i, for $i = 1, 2, 3, 4$ are the exact numbers. Consider the determinant $D = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$ and its floating point approximation $D^* = \begin{vmatrix} a_1^* & a_2^* \\ a_3^* & a_4^* \end{vmatrix}$. Prove that
$$\frac{D^*}{D} \leq e^{4\delta}.$$

2. Let G be a closed subset of $\mathbb{R}$.
   (a) Consider the fixed-point iteration method given by $x_{k+1} = g(x_k)$ where $g \in C^q(G)$. Suppose the iterations converge to a fixed point $z \in G$. Furthermore, assume that q is the first positive integer for which $g^{(q)}(z) \neq 0$ and if $q = 1$ then $|g'(z)| < 1$. Then show that the sequence x_k converges to z with order q.
   (b) Show that the iterative method
$$x_{k+1} = \frac{x_k\,(x_k^2 + 3N)}{3x_k^2 + N}$$
   has third-order convergence for computing $\sqrt{N}$.

3. Richardson method for solving a linear system $Ax = b$ is given by
$$x^{(k+1)} = x^{(k)} + \alpha P^{-1}(b - Ax^{(k)}).$$
   Show that for any nonsingular matrix P, the method is convergent iff
$$\frac{2\texttt{Re}\lambda_i}{\alpha|\lambda_i|^2} > 1, \forall i = 1, \ldots, n$$
   where $\lambda_i \in \mathbb{C}$ are eigenvalues of $P^{-1}A$.

4. Let $P_2(x)$ be the quadratic polynomial interpolating $f(x)$ at x_0, x_1, x_2 with the nodes $x_0 \le x_1 \le x_2$.

   (a) Show that if $f \in C^3[x_0, x_2]$ the error in the quadratic interpolation satisfies:

   $$f(x) - P_2(x) = \frac{1}{6}(x - x_0)(x - x_1)(x - x_2)f'''(\xi(x)), \qquad x_0 \le x \le x_2.$$

   (b) Find the function $\xi(x)$ explicitly for $f(x) = \dfrac{1}{x}$ with $x_0 = 1, x_1 = 2, x_2 = 3$ and $x \neq x_k$ for $k = 0, 1, 2$.

5. Consider the following data arising from an engineering application:

x	0	1	2	3	4
$f(x)$	-1	0	15	80	255

   (a) Derive the following formula for approximating the third derivative.

   $$f'(x) \approx \frac{1}{12h}\left[-f(x + 2h) + 8f(x + h) - 8f(x - h) + f(x - 2h)\right]$$

   (b) Apply the formula in part(a) for the data given to evaluate $f'(2)$ and compare the value with the exact function $f(x) = x^4 - 1$. Explain why the formula in part(a) gives the exact value.

6. Consider a quadrature formula of the type

   $$\int_0^\infty e^{-x} f(x)\, dx = af(0) + bf(c)$$

   (a) Find a, b and c such that the formula is exact for polynomials of the highest degree possible. (Note that $\displaystyle\int_0^\infty e^{-x}\, x^n\, dx = n!$).

   (b) Let $P(x)$ be the polynomial interpolating f at the (simple) point $x = 0$ and double point $x = 2$; i.e. $P(0) = f(0)$, $P(2) = f(2)$ and $P'(2) = f'(2)$. Determine $\displaystyle\int_0^\infty e^{-x}\, P(x)\, dx$ and compare with the result in part (a).

7. Consider the initial value problem

   $$\frac{dy}{dt} = a + by(t) + c\sin y(t), \qquad 0 \le t \le 1$$

   where $y(0) = 1$ and $a, b, c > 0$ are constants. Let us suppose that the solution satisfies $\max\limits_{0\le t\le 1} |y''(t)| = M < \infty$. Consider the approximation $y_{k+1} = y_k + (a + by_k + c\sin(y_k))h$ for $k = 0, 1, 2, \ldots, N - 1$ and let $y_0 = y(0)$ and $h = \dfrac{1}{N}$. Prove that

   $$|y(1) - y_N| \le \frac{Mhe^{b+c}}{2(b + c)}.$$