Introduction to Martin Boundary in Potential Theory

Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane and $\overline{\Omega} \setminus \Omega = \{z \in \mathbb{C} : |z| = 1\}$ its topological boundary. The Poisson kernel $P : \Omega \times (\overline{\Omega} \setminus \Omega) \to \mathbb{R}$ is

$$P(z,w) = \frac{1-|z|^2}{|z-w|^2}.$$

It is used to solve the Dirichlet problem on $\overline{\Omega}$ and also to describe all positive harmonic functions on Ω. A theorem of Herglotz says that for any positive harmonic function on Ω, there exists a unique measure μ on the boundary $\overline{\Omega} \setminus \Omega$ such that

$$f(z) = \int_{\overline{\Omega} \setminus \Omega} P(z,w)\, d\mu(w) \quad (z \in \Omega).$$

If we replace Ω by a general bounded open subset of $\mathbb{R}^N$, it is not true in general that the topological boundary $\overline{\Omega} \setminus \Omega$ is big enough to parameterize a set of harmonic functions which generate all positive harmonic functions on Ω. There is, however, a compactification of Ω called the Martin Compactification having associated boundary Δ for which there is an analogue $M : \Omega \times \Delta \to \mathbb{R}$ of the Poisson kernel such that all positive harmonic functions are of the form

$$f(x) = \int_{\Delta} M(x,y)\, d\mu(y) \quad (x \in \Omega)$$

for some measure μ on the Martin boundary Δ.

In this elementary talk, we describe how to obtain the Martin boundary and give several examples.