Introduction to Martin Boundary in Potential Theory

Let \( \Omega = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the unit disc in the complex plane and \( \overline{\Omega} \setminus \Omega = \{ z \in \mathbb{C} : |z| = 1 \} \) its topological boundary. The Poisson kernel \( P : \Omega \times (\overline{\Omega} \setminus \Omega) \to \mathbb{R} \) is
\[
P(z, w) = \frac{1 - |z|^2}{|z - w|^2}.
\]

It is used to solve the Dirichlet problem on \( \Omega \) and also to describe all positive harmonic functions on \( \Omega \). A theorem of Herglotz says that for any positive harmonic function on \( \Omega \), there exists a unique measure \( \mu \) on the boundary \( \overline{\Omega} \setminus \Omega \) such that
\[
f(z) = \int_{\overline{\Omega} \setminus \Omega} P(z, w) \, d\mu(w) \quad (z \in \Omega).
\]

If we replace \( \Omega \) by a general bounded open subset of \( \mathbb{R}^N \), it is not true in general that the topological boundary \( \overline{\Omega} \setminus \Omega \) is big enough to parameterize a set of harmonic functions which generate all positive harmonic functions on \( \Omega \). There is, however, a compactification of \( \Omega \) called the Martin Compactification having associated boundary \( \Delta \) for which there is an analogue \( M : \Omega \times \Delta \to \mathbb{R} \) of the Poisson kernel such that all positive harmonic functions are of the form
\[
f(x) = \int_{\Delta} M(x, y) \, d\mu(y) \quad (x \in \Omega)
\]
for some measure \( \mu \) on the Martin boundary \( \Delta \).

In this elementary talk, we describe how to obtain the Martin boundary and give several examples.