GMU Applied and Computational Mathematics Seminar
1:30-2:30pm, January 31, 2014, Exploratory Hall, Room 4106

A PDE APPROACH TO NUMERICAL FRACTIONAL DIFFUSION

RICARDO H. NOCHETTO, ENRIQUE OTÁROLA, AND ABNER J. SALGADO

ABSTRACT. We study PDE solution techniques for problems involving fractional powers of symmetric coercive elliptic operators in a bounded domain with Dirichlet boundary conditions. These operators can be realized as the Dirichlet to Neumann map of a degenerate/singular elliptic problem posed on a semi-infinite cylinder, which we analyze in the framework of weighted Sobolev spaces. Motivated by the rapid decay of the solution of this problem, we propose a truncation that is suitable for numerical approximation. We discretize this truncation using first order tensor product finite elements. We derive a priori error estimates in weighted Sobolev spaces, which exhibit optimal regularity but suboptimal order for quasi-uniform meshes and quasi-optimal order-regularity for anisotropic meshes.

We also present some results on efficient computational techniques to solve fractional powers of the fractional Laplace operator: adaptivity and multilevel methods. As a first step towards adaptivity, we present a computable a posteriori error estimator, which exhibits built-in flux equilibration and is equivalent to the energy error up to data oscillation. We design a simple adaptive strategy, which reduces error and data oscillation. We also discuss a $\mathcal{V}$-cycle multilevel method with line smoother, together with a nearly uniform convergence result on anisotropic tensor product discretizations. Numerical experiments reveal a competitive performance of this method.

Finally, to show the flexibility of our approach, we consider the discretization of evolution equations with fractional diffusion and fractional time derivative. We show discrete stability estimates which yield to a novel energy estimate for evolution problems with a fractional time derivative. We consider a first order semi-implicit fully-discrete scheme: first degree tensor product finite elements in space and a first order discretization in time. We present a priori error estimates for the proposed numerical scheme.

Keywords: Fractional diffusion; finite elements; nonlocal operators; degenerate and singular equations; anisotropic elements; evolution problems; Muckenhoupt weights.

Mathematics Subject Classifications (2000): 35S15; 65R20; 65N12; 65N30.

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DEPARTMENT OF MATHEMATICS AND INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA.
E-mail address: `rhn@math.umd.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, USA.
E-mail address: `kike@math.umd.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37996, USA
E-mail address: `asalgad1@utk.edu`