Reconstruction of Shared Nonlinear Dynamics in a Network

Timothy D. Sauer
Department of Mathematical Sciences, George Mason University, Fairfax, Virginia 22030, USA
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We study a general physical network consisting of a collection of response systems with complex nonlinear dynamics, influenced by a common driver. The goal is to reconstruct dynamics, regular or chaotic, that are common to all of the response systems, working from simultaneous time series measured at the responses systems only. A fundamental theorem is stated concerning the reconstruction of the common driver. An algorithm is developed, based on the theorem, to carry out the reconstruction, and is demonstrated with several examples.

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Gaining system information from observed dynamical data is a key problem in many areas of physics. The success of conventional signal processing techniques is due to the great simplification that spectral and related techniques afford when the generating dynamics are predominantly linear. Since linear dynamics can be represented by geometrically simple phase spaces (closed curves and more generally tori for quasiperiodic motions), the question of geometry has traditionally played a minor role in these techniques.

With the recent growth of interest in nonlinear phenomena, geometry has come into the foreground of dynamical data analysis. The early papers of Packard *et al.* [1] and Takens [2] show that deterministic systems can indeed produce dynamics whose invariant set is complex geometrically, and, in particular, fractal. Knowledge of the geometry of the dynamics, recovered from measurements of the dynamics, has been shown to have numerous applications to problems that are otherwise intractable [3]. The capability of these methods for nonlinear systems, built on advanced reconstruction techniques, far exceeds the reach of conventional signal processing.

Takens's Theorem [2] (see also [4] for a formulation more appropriate for fractal attractors) shows how the reconstruction of system geometry can be done from time series [5] measured from the system. This mathematical fact can be successfully applied when the so-called genericity conditions are verified, but this is not always possible. In particular, it is necessary for the observations to be coupled, at least indirectly, to all modes of the dynamics. Many dynamical network topologies fail to meet this condition.

In this Letter we take a first step toward extending the work of Takens *et al.* to analyze networks of nonlinear systems. Our goal is to lay the groundwork for reconstructing the individual dynamical components of coupled networks. Driver-response dynamics, in particular, where one or the other is chaotic, has attracted a great deal of interest recently in electronic circuits [6], structural health monitoring [7], and optics [8], in particular, where various types of synchronization have been detected. Our focus here is on the reconstruction of the driver from measurements taken solely from the response systems, given no prior knowledge of driver or response. We introduce a key theorem that shows how to carry out the reconstruction. The algorithm that implements this theory is described and applied to several examples.

The most basic configuration to be considered is a network consisting of a single system A driving a number of individual response systems $B_1, \dots, B_n$. The problem of reconstructing the dynamics of the driver system is a problem of reconstructing dynamics shared by the response systems. Working only from time series output, there can be a natural ambiguity about where the shared dynamics lies, whether in the driver or in identical subsystems of the response systems. In some cases it will be impossible to disambiguate the two possibilities, and we use the term "shared dynamics" for this reason. For many modeling purposes, the difference is benign.

There are two principal motivations for reconstructing shared dynamics. First, for some applications, uncovering the dynamics of the common source may be the primary goal. This is akin to inverting a filter bank, where the filters are nonlinear dynamical systems in their own right. Second, prediction and filtering tasks for each response system can be done more accurately with extra system information. In other words, information from one or several secondary response signals, driven by the same source, can lead to significantly improved analysis and geometric reconstruction of the signal from the response system of interest.

This work depends on a reconstruction theorem for skew products due to Stark [9] (see also Casdagli [10]). That theorem states conditions under which the state space of a system consisting of the combined driver and response can be reconstructed. In the present article, the goal is to use the output of several response units to separate out the dynamics of the driver from the rest of the dynamics, as a way of breaking the system down into its component parts.

Let $f: A \to A$ represent a continuous, ergodic dynamical map on a compact attractor A, and for $i = 1, \dots, n$, denote by B_i a dynamical system driven by A. Denote the driving by $g_i: A \times B_i \to B_i$, and require $g_i(a, \cdot)$ continuous for each $a \in A$. The map g_i is called a skew product. Iterated dynamics is denoted by recursively defining $g_i^{(k)}: A \times B_i \to B_i$ as $g_i^{(k+1)}(a, b) = g_i(f^k(a), g_i^{(k)}(a, b))$. Consider an attractor Z of the product dynamics in $A \times B_1 \times \ldots \times B_n$, and denote by $Z_i = \Pi_i(Z)$ the projection of Z into $A \times B_i$. Thus Z_i represents the attractor of the combined dynamics in $A \times B_i$.

Assume $h_i: B_i \to R$ are real-valued continuous maps for $i = 1, \ldots, n$, interpreted as observation functions. The values of h_i can be viewed as a time series measured from B_i. The n simultaneous time series, as in Fig. 1, can be used to construct delay coordinates. For each i the so-called delay map can be written as

$$\Phi_i(a, b) = [h_i(g_i^{(0)}(a, b)), \ldots, h_i(g_i^{(m-1)}(a, b))].$$

The skew product reconstruction theorem of Stark [9] holds that under mild technical conditions for generic g_i and h_i, the delay map Φ_i embeds the product $A \times B_i$ into R^m, for sufficiently large m. In particular, there is a one-to-one correspondence from Z_i to its image $X_i \equiv \Phi_i(Z_i)$, for each i. In particular, each reconstructed x_i in X_i is associated to a unique driver state a in A.

This simple observation motivates the reconstruction of the shared dynamics of the B_i. Let $\phi_i = \Phi_i \Pi_i$ denote the ith component of the reconstruction of the attractor Z, and consider two points $z = (a, b_1, \ldots, b_n)$ and $z' = (a', b'_1, \ldots, b'_n)$ in Z. The points z and z' represent two states of the entire system $A \times B_1 \times \ldots \times B_n$. Assume that the ith coordinate of the reconstruction agrees at z and z', that is $\phi_i(z) = \phi_i(z')$. Then $\Pi_i(z) = \Pi_i(z')$, because Φ_i is one-to-one due to the skew product reconstruction theorem. Therefore $a = a'$, meaning that the driver states of z and z' agree. Note that this information is novel when applied to the other subsystems $j \neq i$.

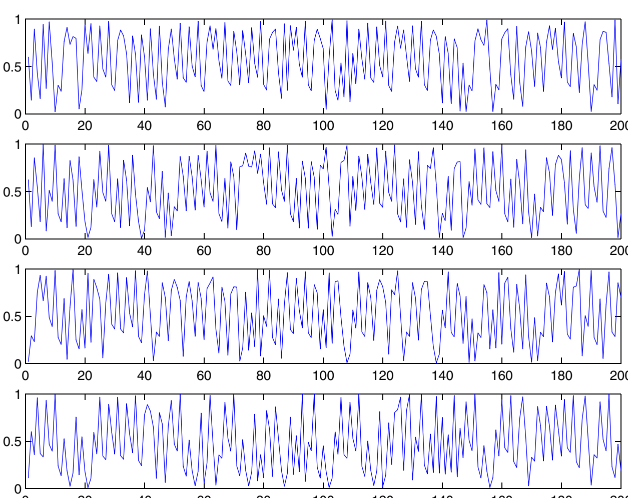
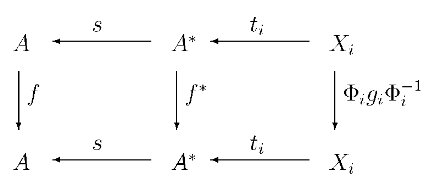


FIG. 1 (color online). Four time series representing output of $B_1, \ldots, B_4$ from (4). Each B_i is driven by an identical input signal from a period 3 attractor A given in Eq. (3).

This leads to a classification scheme for the set of points in the reconstructed attractor X_i of the B_i dynamics. Fix one i arbitrarily. Define A^* to be the set of equivalence classes of reconstructed points in X_i, generated by the relation $x_i = \phi_i(z) \sim x'_i = \phi_i(z')$ if there exists a $j \neq i$, $1 \leq j \leq n$ and x_j such that $\phi_j(z) = \phi_j(z')$. In other words, identical returns in component j of the reconstruction implies an identical drive state, so the corresponding component i should be grouped in order to reproduce the drive. Each set of equivalent points of the fixed X_i, under this relation, is considered to be a unique point in A^*. It is clear that the construction of A^* is independent of the choice of i, in the sense that any other choice would result in a set of equivalence classes A^* that is in one-to-one correspondence with any other choice.

There are three functions that can be defined for the set A^*. First, every equivalence class a^* in A^* has associated with it a unique a in A, so define the function s from A^* to A by $s(a^*) = a$. The function s is *onto*, meaning that the image of s is all of A. (This follows from the fact that the dynamics f is ergodic on A.) Second, there is a natural dynamical rule f^* from the set A^* to itself that is inherited from the dynamics on the delay coordinates. The equivalence class $f^*(a^*)$ is defined to be the one the elements of a^* are mapped to under the system dynamics g_1. In addition to the functions s and f^*, for each $1 \leq i \leq n$, the function t_i from X_i to A^* can be defined by sending each $x_i \in X_i$ to the equivalence class of x_i.

The following diagram shows the relation between the functions t_i, s and the new dynamical system f^* on A^*. Because f^* is onto, the right half of the diagram shows that f^* satisfies the definition of semiconjugacy. The map $f^*: A^* \to A^*$ is said to be *semiconjugate* to the map $f: A \to A$ if there exists an onto map $s: A^* \to A$ satisfying $f \circ s = s \circ f^*$, that is, the left side of the diagram "commutes"



The analogous statement about the right side of the diagram is also true. This is summarized in the following theorem.

Shared dynamics reconstruction theorem.—Assume A has ergodic dynamics, and in addition drives B_i, $1 \leq i \leq n$. Choose m large enough and f, g_i generic such that all skew products $A \times B_i$ are reconstructed in R^m. Define $f^*: A^* \to A^*$ as above. Then, under genericity conditions, (1) the map f^* is semiconjugate to f, and (2) for each i, the map $\Phi_i g_i \Phi_i^{-1}$ is semiconjugate to f^*.

Roughly speaking, if f_1 is semiconjugate to f_2, then f_1 "contains" the dynamics of f_2. The content of the theorem is that according to the left-hand square of the above diagram, A^* captures at least the dynamics of the driver

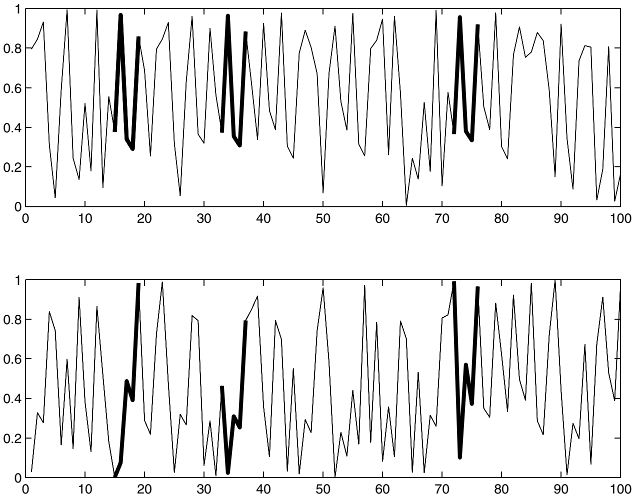


FIG. 2. Algorithm finds delay-coordinate points from a small ε neighborhood in the B_j reconstruction (upper trace, in bold), and collects together the corresponding simultaneous points in each B_i reconstruction, $i \neq j$ (lower trace, in bold), independent of their proximity in B_i.

A, and may contain more. However, according to the right side of the diagram for each i, any extra dynamics in A^* must be common to all of the X_i, due to part (2) of the theorem. This is the meaning of "shared dynamics."

For example, if all B_i have an identical subsystem that is moving synchronously, independent of the driver A, this will also be reconstructed by A^*. Alternatively, if part of the driver's dynamics is considered part of each B_i, that will be captured as well by A^*. This reflects the fact that the separation between driver and response depends on the definition of the A and the B_i. The theorem says only that (1) all of A must occur in A^*, and (2) all dynamics in A^* must be shared by all B_i.

Next we show how this theorem leads to an algorithm that extracts shared dynamics of the B_i from time series data observed and reconstructed in the X_i.

Shared Dynamics Algorithm.—Choose m large enough to unfold the dynamics on each $A \times B_i$, and use delay coordinates to create the reconstructed attractor X_i, which is in one-to-one correspondence with Z_i. Choose one of the B_i arbitrarily, say B_1. The basis of the algorithm is to group together points in B_1 that lie over the same point in A, the so-called fibers over A. According to the theorem, at each time t when the dynamics returns to the same point in B_j, the value of a in A is the same. Figure 2 shows three nearly identical delay vectors in X_j. Although at the corresponding times, the vectors in X_i for $i \neq j$ look unconnected, they must in fact be different points lying over (approximately) the same a. With this information, one can search for delay vectors in X_j that are close in the delay reconstruction, and return information to X_1 about points over the same a. A neighborhood size ε must be chosen to decide the meaning of nearly identical, for this purpose. The degree of discretization of the resulting dynamical attractor A^* will depend on this choice.

Using this method of associating points to the fibers over A, one proceeds through all points of the reconstructed attractor X_1 to fit them in an appropriate equivalence class. Choosing representatives for the equivalence classes from a chain of overlapping ε neighborhoods retains the topological form imposed by the original dynamics. Note that no reembedding is necessary, since the points of A^* constitute a subset of the reconstructed X_1, which has no self-intersections by assumption.

We demonstrate first with a simple example. Let A be a circle with irrational rotation dynamics driving two discrete response systems, a period-two orbit in B_1 and a period-three orbit in B_2. Under additive driving, the attractors Z_1 and Z_2 will typically be unions of two and three topological circles, respectively. The above theorem implies that the reconstructed set A^* is semiconjugate to an irrational rotation on A, and Z_1 and Z_2 are individually semiconjugate to A^*. Since Z_1 and Z_2 have no dynamics in common except for the drive, A^* must be an irrational rotation on a circle. In this case, the driver dynamics are reconstructed completely: $A \equiv A^*$.

The shared dynamics algorithm was applied to this problem, where A is an irrational rotation $x \rightarrow x + \omega$ of approximately 0.06 Hz on a circle, B_1 is the logistic map with period-two dynamics

$$y \rightarrow a_1 y(1 - y) + b_1, \tag{1}$$

where $a_1 = 3.2$, $b_1 = 0.1$, and B_2 is the circle map

$$y \rightarrow a_2 \sin 6\pi y + y + 1/3 + b_2 x (\mathrm{mod} 1), \tag{2}$$

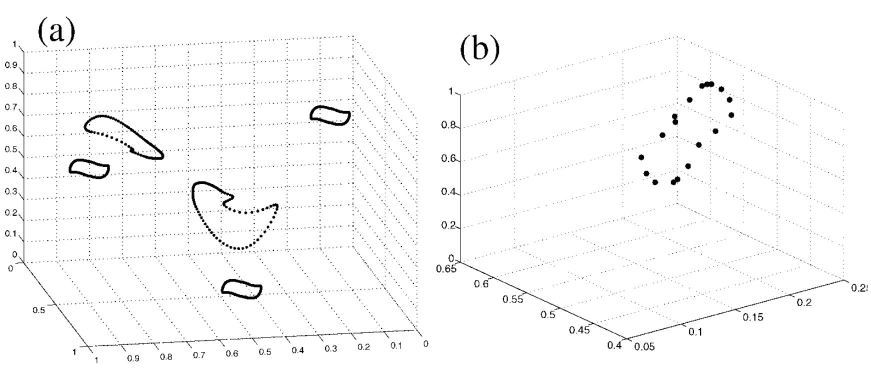


FIG. 3. Shared dynamics algorithm for system (1) and (2). (a) Union of reconstructed attractors X_1 and X_2. (b) The set A^* determined by the algorithm exhibits a circle rotation.

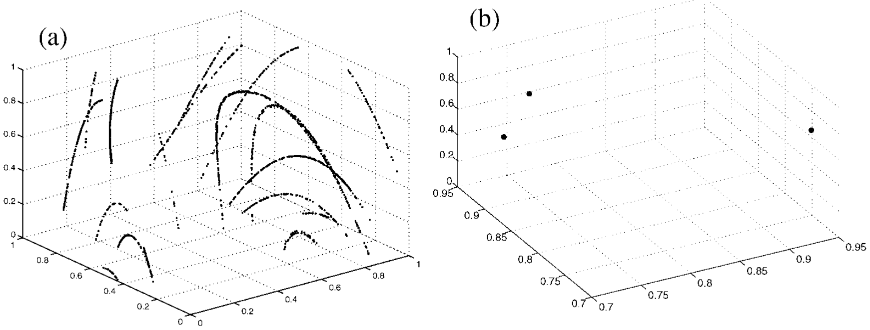


FIG. 4. Shared dynamics algorithm for system (3) and (4). (a) Reconstruction of X_1 shown; X_2, X_3, X_4 are similar in appearance. (b) The set A^* determined by the algorithm agrees with the period-three dynamics of the driver A.

where $a_2 = b_2 = 0.1$, undergoing period-three dynamics. The attractors X_1 and X_2 were reconstructed from a length 1 K time series, using $h_1 = h_2$ to be the identity functions, and are displayed together in Fig. 3(a). The output A^* of the algorithm is shown in Fig. 3(b). The set A^* comprises equivalence classes based on the part of X_1 at upper right in Fig. 3(a). The topology of the driver A, a circle, is reconstructed up to the discretization enforced by the neighborhood size ε set in the algorithm. Although quasiperiodic circle dynamics is used for the sake of clarity in this example, driving with fractal chaotic attractors yields similar results (not shown).

A second example shows that the algorithm successfully discriminates shared behavior even when the B_i are nearly identical but different systems. In addition, the example shows that continuity in the dynamics is not essential. Define the driver A to be the logistic map in the period-three window

$$x \longrightarrow \lambda x(1 - x), \tag{3}$$

with $\lambda = 3.835$, and let B_i, $i = 1, \ldots, 4$ be discontinuous logistic maps

$$y \longrightarrow a_i y(1 - y) + b_i x(\mathrm{mod}1), \tag{4}$$

where $a_1 = 3.81, a_2 = 3.82, a_3 = 3.83, a_4 = 3.84$, and $b_i = 0.45$ for all $1 \leq i \leq 4$. Time series from the B_i, using the identity function for h_i, are shown in Fig. 1.

Figure 4(a) shows the skew dynamics of B_1, reconstructed from a time series of length 1 K. The remaining B_i are not shown, but bear great similarity to B_1. Despite the similarity, the algorithm is able to correctly group all fibers over points in the driver A. Figure 4(b) shows that $A^* \equiv A$, which is the correct answer.

The efficiency of reconstructing the driver dynamics from the response signals grows with the length of time series, as the neighborhood size ε in the algorithm can be decreased accordingly. For example, the A^* in Fig. 3(b) can be extracted with greater resolution using a larger data set. Efficiency also depends on the heterogeneity of the B_i. The more alike the B_i dynamics, as in Eq. (4), the more difficult the discrimination between fibers over A, and the more data will be required.

Nonlinear and chaotic dynamical systems are known for emergent properties and other obstructions to reductionist analysis. We do not expect to develop a theory of nonlinear transfer functions analogous to the relatively well-understood linear case. However, it is our hope that the theorem and associated algorithms presented here spur the development of methodologies to break down nonlinear systems into smaller, simpler parts where possible. This kind of nonlinear network analysis will be useful whenever less than full connectivity exists, a scenario that essentially contradicts the assumptions made in Takens's theory. In addition to physical systems, many biological systems where "wiring diagrams" are known to connect important subsystems may provide fertile application areas for this type of analysis.

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