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Observing periodically forced systems of difference equations

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Dedicated to Peter Kloeden on the Occasion of his 60th Birthday

We discuss the problem of determining the periodicity of the common driver of a system of periodically forced difference equations. Methods from topological reconstruction of attractors are applied to build a semiconjugacy to a topological version of the driver. A computational algorithm built from the semiconjugacy uses observations of the equation trajectories to isolate the driver period.

Keywords: difference equation; nonautonomous; periodicity; driver dynamics

1. Introduction

Extracting information from dynamical data tends to be difficult when the underlying systems have nonlinearities. Although reconstruction of autonomous systems from data has been discussed frequently in the literature [1,4–7,10], little attention has been paid to the nonautonomous case. In this article, we discuss a method of drawing conclusions about the dynamical content of a driver that simultaneously influences a system of difference equations.

Consider the system of difference equations of form:

$$\begin{aligned} x_t^1 &= f^1(x_{t-1}^1, \dots, x_{t-n}^1, d_t), \\ &\vdots \\ x_t^k &= f^k(x_{t-1}^k, \dots, x_{t-n}^k, d_t), \end{aligned} \tag{1}$$

where $f^1, \dots, f^k$ are continuously differentiable functions and $\{d_t\}$ is a periodic sequence, say $d_t = d_{t+p}$ for some positive integer p, the period of the sequence. For the purposes of our discussion, we will assume that the equations $f^1, \dots, f^k$ are unknown to us, and that we can only observe the outputs $x_t^1, \dots, x_t^k$ as functions of time. Our goal is to find p, the period of the driving sequence $\{d_t\}$.

Takens' theorem [10] discusses conditions under which observations allow a topologically accurate reconstruction of the complete system dynamics. However, such a reconstruction does not in general distinguish the driver sequence from the rest of the system. Our present goal goes beyond what is promised by Takens' theorem, and isolate the influence of the driving sequence. Moreover, if the theory can be developed, it is reasonable to expect that the goal

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can be achieved with a far smaller data requirement than needed to reconstruct the entire system dynamics.

Two recent extensions of Takens' theorem are relevant to our development. Stark [9] proved a reconstruction theorem for skew product systems that was used in [8] to develop a driver reconstruction theorem. An algorithm based on this theorem is discussed below, along with an illustrative example. In the example, a set of five second-order chaotic Henon maps are driven by a period-seven sequence. The output of the five maps is used (without knowledge of the maps generating the output) to infer the periodicity of the driver.

The goal of this article is to further develop the computational techniques for system identification applied to network models of difference equations. The joint periodicity question presents itself rather generally, whenever simultaneous driving of multiple, observable systems occurs.

2. Background and theoretical results

We recall a reconstruction theorem for skew products due to Stark [9] (see also Casdagli [2]).

THEOREM 2.1 (STARK [9]). *Let D and X be compact manifolds,* $\dim(D) = d$, $\dim(X) = n \geq 1$. *Let $m \geq 2d + 2n + 1$, and assume the periodic orbits of period $< 2m$ of $g : D \to D$ are isolated and have distinct eigenvalues. Then there exists an open, dense set of C^1 functions $f : D \times X \to X$ and $h : X \to R$ for which the m-dimensional delay map*

$$(d_t, x_t) \to [h(f(d_t, x_t)), \ldots, h(f(d_{t+m-1}, x_{t+m-1}))]$$

is an embedding of $D \times X$ into R^m.

Theorem 2.1 states conditions under which the state space of a system consisting of the combined driver and response can be reconstructed. Our goal in this article is to use the output of several response units to separate out the dynamics of the driver from the rest of the dynamics, as a first step to breaking the system down into its component parts.

The state spaces of the k difference equations in (1) will be referred to as $X_1, \ldots, X_k$. Apply Stark's theorem to each $D \times X_i$ individually, where D represents the periodic dynamical system of the driving system and X_i is the state space of the ith nonautonomous system. Consider the 'observation function' $h_i : X_i \to R$ which projects onto the first coordinate. According to Theorem 2.1, for generic dynamics and sufficiently large m, there is a one-to-one correspondence between m-tuples $(x^i_t, \ldots, x^i_{t+m-1})$ and states of the dynamical invariant set A_i in $D \times X_i$. The set of m-tuples of form $(x_t, \ldots, x_{t+m-1})$ is usually called the delay-coordinate representation, a common tool in attractor reconstruction research.

Our use of delay-coordinates in the current context is the following important fact: as a consequence of the one-to-one correspondence guaranteed by Theorem 2.1, the equality

$$(x^i_t, \ldots, x^i_{t-m+1}) = (x^i_{t'}, \ldots, x^i_{t'-m+1}) \tag{2}$$

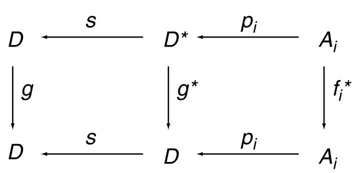
implies that $d_t = d_{t'}$. This fact is exploited below to identify individual states of the driver.

Due to continuity, exact equality in (2) is not essential for information to be extracted. If the difference is small, say in the Euclidean norm, then d_t and $d_{t'}$ must also be near.

Therefore, it is important to choose m large enough so that Stark's theorem applies to $D \times X_i$ for $i = 1, \ldots, k$. We choose some $\epsilon > 0$ and for a given $(x_t^i, \ldots, x_{t-m+1}^i)$, group the set of times t' for which (2) holds within ϵ. In this way, we form a set of equivalence classes.

The identification of the equivalence classes is simplified by choosing an arbitrary X_i, and working in its delay coordinate space. This space contains a one-to-one representation of the attractor A_i in $D \times X_i$, according to Stark's theorem. We will denote by D^* the set of equivalence classes, representations of a quotient space of A_i. We will see next that D^* is a semiconjugacy with the dynamics of D.

To simplify discussion of the semiconjugacy, we draw a commutative diagram. Three functions can be defined for the set D^* that form links in the diagram. First, every equivalence class d^* in D^* has associated with it a unique d in D, so define the function s from D^* to D by $s(d^*) = d$. The function s is *onto*, meaning that the image of s is all of D. (This follows from the fact that the dynamics g, being a finite periodic orbit, takes on all states of D.) Second, there is a natural dynamical rule g^* from the set D^* to itself that is inherited from the dynamics on the delay coordinates. The equivalence class $g^*(d^*)$ is defined to be the one the elements of d^* are mapped to under the system dynamics f_1. In addition to the functions s and g^*, for each $1 \leq i \leq k$, the function p_i from A_i to D^* can be defined by sending each $a_i \in A_i$ to the equivalence class of a_i. The following diagram shows the relation between the functions p_i, s and the new dynamical system g^* on D^*.



Because g^* is onto, the right half of the diagram shows that g^* satisfies the definition of semiconjugacy. The map $g^* : D^* \to D^*$ is said to be *semiconjugate* to the map $f : D \to D$ if there exists an onto map $s : D^* \to D$ satisfying $g \circ s = s \circ g^*$, that is, the left side of the diagram commutes. The analogous statement about the right side of the diagram is also true. The following theorem was introduced in [8]:

THEOREM 2.2 (SHARED DYNAMICS RECONSTRUCTION THEOREM). *Assume $g : D \to D$ is ergodic, and in addition drives X_i, $1 \leq i \leq k$ as in* (1). *Choose m large enough and g, f^i generic such that all skew products $D \times X_i$ are reconstructed in R^m. Define $g^* : D^* \to D^*$ as the map induced as above. Then, generically,* (1) *the map g^* is semiconjugate to g, and* (2) *for each i, the induced map is semiconjugate to g^*.*

Roughly speaking, if g_1 is semiconjugate to g_2, then g_1 'contains' the dynamics of g_2. The content of the theorem is that according to the left hand square of the above diagram, D^* captures at least the dynamics of the driver D, and may contain more. However, according to the right side of the diagram for each i, any extra dynamics in D^* must be common to all of the X_i, due to part (2) of the theorem. This is the meaning of 'shared dynamics'.

3. Shared dynamics algorithm

Next, we show how this theorem leads to a computational method that extracts shared dynamics of the D using time series data observed from the X_i.

Shared dynamics algorithm. Choose m large enough to unfold the dynamics on each $D \times X_i$, and use delay coordinates to create the reconstructed attractor A_i, which is in one-to-one correspondence with $D \times X_i$. Choose one of the X_i arbitrarily, say X_1. The basis of the algorithm is to group together points in X_1 that lie over the same point in D, the so-called fibres over D. According to the theorem, at each time t when the dynamics returns to the same point in X_j, the state of d in D is the same. With this information, one can search for delay vectors in X_j that are close in the delay reconstruction, and return information to X_1 about points over the same driver state d. A neighbourhood size ε must be chosen to decide the meaning of nearly identical, for this purpose. The degree of discretization of the resulting dynamical attractor D^* will depend on this choice.

Using this method of associating points to the fibres over D, one proceeds through all points of the reconstructed attractor X_1 to fit them in an appropriate equivalence class. Choosing representatives for the equivalence classes from a chain of overlapping ε-neighbourhoods retains the topological form imposed by the original dynamics. Note that no re-embedding is necessary, since the points of D^* constitute a subset of the reconstructed X_1, which has no self-intersections by assumption.

4. Implementation of algorithm

In this section, we illustrate the use of the shared dynamics algorithm on an illustrative example. A system of difference equations of the form given in (1) is constructed and the output data x_i are provided to the algorithm for analysis.

Example. Consider the system of forced nonlinear second-order difference equations suggested by Henon [3]:

$$x_{t+1}^j = a^j - (x_t^j)^2 + b^j x_{t-1}^j + c^j d_t, \tag{3}$$

where $j = 1, \ldots, 5$. The parameters a^j and b^j are chosen randomly from the interval $[1.26, 1.28]$ and $[0.2995, 0.3005]$, respectively, and all coupling parameters are set to $c^j = 0.07$. Thus five versions of the difference equation are driven by the periodic sequence $\{d_t\}$. The sequence is defined as $d_t = \sin 2\pi t/p$, where $p = 7$ in this numerical experiment.

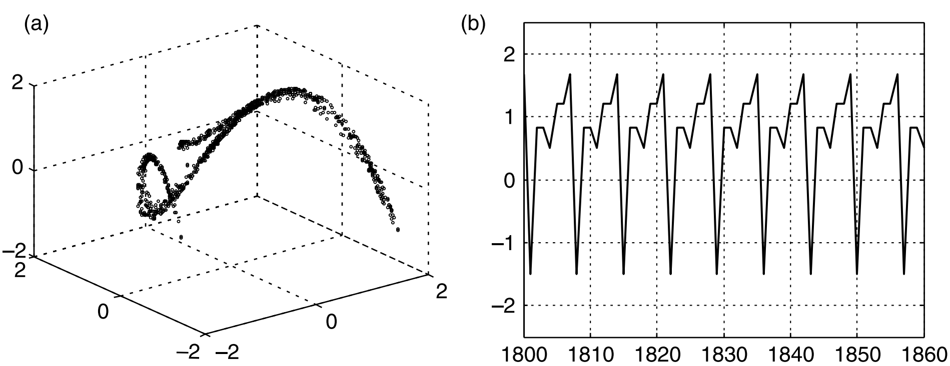


Figure 1. (a) Delay coordinate reconstruction of one of the five skew products from (3). (b) A representative section of the reconstructed driver dynamics that is output from the algorithm. The periodicity of length seven is apparent.

We now apply the shared dynamics algorithm to data recorded from the first five equations only. Thus five-tuples $(x_t^1, \ldots, x_t^5)$ form a chaotic multivariate time series that is observed and used as input to the algorithm as explained in the previous section.

Figure 1(a) shows the delay coordinate embedding of the reconstructed attractor A_1 of $D \times X_1$, using the embedding dimension $m = 3$. A trajectory of length 2000 time units is shown. The result of the shared dynamics algorithm with $\epsilon = 0.007$, the reconstructed driver D^*, is shown in Figure 1(b). Each point in the time series of Figure 1(b) represents an equivalence class of delay coordinate vectors. The period-seven sequence $\{d_t\}$ is recovered, as desired.

5. Conclusions

This article demonstrates a type of signal processing for difference equations, the idea being to identify system characteristics from output data. The specific goal in this case is to identify the dynamics of the common driver in a system of nonautonomous difference equations.

The example provided deals with a periodic sequence driving five logistic maps. The algorithm is not limited to periodic drivers, although the job of identifying the result will be more difficult the more complicated the driver dynamics. If the driver is chaotic, for example, sophisticated system identification methods may be needed to analyze and classify the result.

In the case of a chaotic system driving other chaotic systems, for example, although the algorithm introduced here will work in principle, the data requirements may be challenging. For periodic driving, we have shown in the example that a few thousand data points suffice to determine the driver.

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