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Global convergence of max-type equations

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Consider a difference equation whose evolution rule is defined as the maximum of several first-order equations. It is shown that if the first-order equations are individually contractive, then the aggregated max-type equation converges to a fixed point. A corresponding result holds for local convergence.

Keywords: difference equation; maximum; asymptotic convergence; global convergence; stability

1. Introduction

Let p be a positive integer and let $f_i : R \to R$ for $i = 1, \ldots, p$ be real-valued functions. Given the initial data $x_1, \ldots, x_p$, we define the max-type difference equation

$$x_n = \max\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\}. \tag{1}$$

DEFINITION 1.1. *The function f is called* contractive *if there exists* $0 \le \alpha < 1$ *and a real number r such that* $|f(x) - r| \le \alpha |x - r|$ *for all x.*

DEFINITION 1.2. *The solution* $\{x_n\}_{n=1}^{\infty}$ *of a difference equation is called* globally convergent *if there exists r such that for every set of initial values,* $\lim_{n\to\infty} x_n = r$. *In this case, the equilibrium r is called* globally attractive.

We will show that if the f_i are contractive with fixed points r_i, then the difference equation (1) is globally convergent, or more precisely, converges in the limit to $\max\{r_i\}$ for any set $\{x_1, \ldots, x_p\}$ of initial values. As an example, consider the difference equation

$$x_n = \max\left\{A_1 x_{n-1}^{\alpha_1}, \ldots, A_p x_{n-p}^{\alpha_p}\right\}, \tag{2}$$

where the $A_i > 0$ and $-1 < \alpha_i < 1$ for $i = 1, \ldots, p$. By a logarithmic change of coordinates, equation (2) is converted to a difference equation of type (1), and it can be concluded from Corollary 2.4 that (2) converges to $\max A_i^{1/(1-\alpha_i)}$ for all positive initial conditions.

Max-type difference equations have been considered by a number of authors, including [1–9,11–14]. Ladas poses an interesting array of problems in [9]. Periodic and

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more complicated behaviour is typical when the contractive hypothesis does not hold. In this article, we gather together general situations when contractivity of individual first-order components translates to convergence of the aggregated difference equation with maximum.

Theorem 2.3 below is the main global convergence result, proved in a context slightly more general than (1). The techniques used to prove Theorem 2.3 can also be applied to prove a local convergence version, Theorem 3.2.

2. Global convergence

The following two lemmas provide the facts needed to prove the main results.

LEMMA 2.1. *Let p be a positive integer, r and* $0 \le \alpha < 1$ *real numbers, and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Assume that for each n there exists i, possibly depending on n,* $1 \le i \le p$, *such that* $|x_n - r| \le \alpha |x_{n-i} - r|$. *Then* $\lim_{n\to\infty} x_n = r$.

Proof. For each positive integer j, consider

$$M_j = \max_{1\le i\le p} |x_{jp+1-i} - r|.$$

It suffices to show that $M_{j+1} \le \alpha M_j$ for all j. To do this, we show $|x_{jp+k} - r| \le \alpha M_j$ for $1 \le k \le p$ by induction.

For $k = 1$, $|x_{jp+1} - r| \le \alpha |x_{jp+1-i} - r|$ for some $1 \le i \le p$ by hypothesis, and so $|x_{jp+1} - r| \le \alpha M_j$. For $1 < k \le p$, there exists $1 \le i \le p$ such that

$$\begin{aligned} |x_{jp+k} - r| &\le \alpha |x_{jp+k-i} - r| \\ &\le \alpha \max\{\max_{1\le m\le k-1} |x_{jp+m} - r|, \max_{0\le m\le p-k} |x_{jp-m} - r|\} \\ &\le \alpha \max\{\alpha M_j, M_j\} \le \alpha M_j, \end{aligned}$$

completing the induction argument.

It follows immediately that $M_{j+1} \le \alpha M_j$, and so $\lim_{j\to\infty} M_j = 0$. □

LEMMA 2.2. *Let* u_1, u_2, $y_1 \le y_2$ *and* $s_2 \le s_1$ *be real numbers, and assume* $|y_i - s_i| \le \alpha |u_i - s_i|$ *for some* $0 \le \alpha < 1$ *and* $i = 1, 2$. *Then* $|y_2 - s_1| \le \alpha |u_j - s_1|$ *for* $j = 1$ *or* $j = 2$.

Proof. If $y_2 \le s_1$, then $|y_2 - s_1| = s_1 - y_2 \le s_1 - y_1 = |s_1 - y_1| \le \alpha |u_1 - s_1|$, so the conclusion is proved with $j = 1$.

We may henceforth assume that $s_1 < y_2$. Note that either $u_2 < s_2$ or $u_2 > s_1$, for if $s_2 \le u_2 \le s_1$, then

$$y_2 - s_2 = |y_2 - s_2| \le \alpha |u_2 - s_2| = \alpha(u_2 - s_2) \le \alpha(s_1 - s_2) < \alpha(y_2 - s_2),$$

a contradiction. There are two remaining cases. □

Case 1. $s_1 < y_2$ and $u_2 < s_2$.

Since $|y_2 - s_2| = y_2 - s_2 \geq y_2 - s_1$, it follows that

$$\begin{aligned}|y_2 - s_1| &= y_2 - s_1 \leq y_2 - s_2 = |y_2 - s_2| \\ &\leq \alpha|u_2 - s_2| = \alpha(s_2 - u_2) \leq \alpha(s_1 - u_2) = \alpha|s_1 - u_2|,\end{aligned}$$

and we may set $j = 2$.

Case 2. $s_1 < y_2$ and $s_1 < u_2$.

In this case,

$$\begin{aligned}|y_2 - s_1| &= y_2 - s_1 = y_2 - s_2 + s_2 - s_1 = |y_2 - s_2| + s_2 - s_1 \\ &\leq \alpha|u_2 - s_2| + s_2 - s_1 = \alpha(u_2 - s_2) + s_2 - s_1 \\ &= \alpha(u_2 - s_1) + (\alpha - 1)(s_1 - s_2) \\ &\leq \alpha(u_2 - s_1) = \alpha|u_2 - s_1|,\end{aligned}$$

so we may set $j = 2$, completing the proof. □

THEOREM 2.3. *Consider p nonnegative integers* $q_1, \ldots, q_p$, *and let* $0 \leq \alpha < 1$. *Assume for each i, j satisfying* $1 \leq i \leq p, 1 \leq j \leq q_i$ *there exists a function* $f_{ij} : R \rightarrow R$ *and a real number* r_{ij} *satisfying*

$$|f_{ij}(x) - r_{ij}| \leq \alpha|x - r_{ij}|,$$

for all x. Then for any set $\{x_1, \ldots, x_p\}$ *of initial values, the solution of the difference equation*

$$x_n = \max_{1 \leq i \leq p, 1 \leq j \leq q_i} \{f_{ij}(x_{n-i})\}, \tag{3}$$

converges to $\max_{i,j} r_{ij}$.

Proof. We will use Lemma 2.1 where $r = r_{i_m j_m} = \max_{i,j} r_{ij}$. For each n, choose i', j' such that $\max_{i,j}\{f_{ij}(x_{n-i})\} = f_{i'j'}(x_{n-i'})$. Next, apply Lemma 2.2 with $u_1 = x_{n-i_m}, y_1 = f_{i_m j_m}(x_{n-i_m}), u_2 = x_{n-i'}, y_2 = f_{i'j'}(x_{n-i'}), s_1 = r_{i_m j_m}$ and $s_2 = r_{i'j'}$. Lemma 2.2 implies that

$$|x_n - r_{i_m j_m}| = |\max_{i,j}\{f_{ij}(x_{n-i})\} - r_{i_m j_m}| \leq \alpha|z - r_{i_m j_m}|,$$

where $z = x_{n-i_m}$ or $x_{n-i'}$. This satisfies the hypotheses of Lemma 2.1, so

$$\lim_{n \rightarrow \infty} x_n = r_{i_m j_m}.$$

Setting all $q_i = 1$ in Theorem 2.3 covers the special case referred to as equation (1) in the introduction.

COROLLARY 2.4. *Let $r_1, \ldots, r_p$ be real numbers and assume $f_i : R \to R$ for $i = 1, \ldots, p$ satisfy $|f_i(x) - r_i| \le \alpha|x - r_i|$ for all x, where $0 \le \alpha < 1$. Then for any set $\{x_1, \ldots, x_p\}$ of initial values, the solution of difference equation*

$$x_n = \max\{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\}, \tag{4}$$

converges to $\max_i r_i$ *as* $n \to \infty$.

Example 2.5. It follows from Corollary 2.4 that the difference equation

$$x_n = \max\left\{\frac{1}{a_1 + x_{n-1}^2}, \ldots, \frac{1}{a_p + x_{n-p}^2}\right\}, \tag{5}$$

where $a_i > 3/4$ for $i = 1, \ldots, p$ is globally convergent.

In fact, one can check that the first derivative of $f(x) = 1/(a + x^2)$ is always smaller than 1 in absolute value if $a > 3/4$, so the mean value theorem implies that the hypotheses of Corollary 2.4 are satisfied when r_i denotes the real root of the equation $x^3 + a_i x = 1$. The root r_i lies between 0 and 1, and is a decreasing function of a_i. Therefore, Corollary 2.4 says that for any initial values $\{x_1, \ldots, x_p\}$, the solution $\{x_n\}_{n=1}^{\infty}$ of (5) is convergent to the real root r_{i_m} of the equation $x^3 + a_{i_m} x = 1$, where the i_m is the integer satisfying $a_{i_m} = \min_{1 \le i \le p}\{a_i\}$.

Example 2.6. In [13], Sun considers the difference equation

$$x_n = \max\left\{A_1 x_{n-1}^{\alpha_1}, \ldots, A_p x_{n-p}^{\alpha_p}\right\}, \tag{6}$$

where $A_i > 0, -1 < \alpha_i < 0$ for $i = 1, \ldots, p$ and $x_1, \ldots, x_p > 0$ are initial values. Sun proves that positive solutions are globally convergent if $p = 2$, and conjectures that the same holds for $p > 2$.

The conjecture is proved in [12]. An explicit proof is given there for $p = 3$, along with the comment that the proof for general p is only technically complicated. One can view our contribution in this article as straightening out these technical complications.

We will also enlarge the range of the α_i by assuming $-1 < \alpha_i < 1$ and let p be an arbitrary positive integer. Set $y_n = \log x_n$ in (6). In the new coordinates, the ith equation is $y_n = \alpha_i y_{n-i} + \log A_i$ and due to monotonicity of the logarithm, (6) is replaced with

$$y_n = \max\{\alpha_1 y_{n-1} + \log A_1, \ldots, \alpha_p y_{n-p} + \log A_p\}.$$

For any set of positive initial values $x_1, \ldots, x_p$, the y_n sequence converges to the maximum $\log A_i/(1 - \alpha_i)$, so that

$$\lim_{n\to\infty} x_n = \max_{1 \le i \le p} A_i^{1/(1-\alpha_i)}.$$

This proves convergence of (6) for $-1 < \alpha_i < 1, A_i > 0$, and for all positive initial conditions.

Example 2.7. The hypothesis $|\alpha_i| < 1$ of Theorem 2.3 is necessary. For example, the equation

$$x_n = \max\{-x_{n-1}, x_{n-2}\}, \tag{7}$$

has nonconvergent solution $\{-1, 1, -1, 1, \ldots\}$. Much more varied dynamics follows when the contractiveness hypothesis is relaxed, as discussed in Ladas [9] and references therein.

Another special case of Theorem 2.3 is the following, where we set $p = 1$.

Corollary 2.8. *Let q be a positive integer, and consider real numbers r_j and functions $f_j : R \to R$ for $j = 1, \ldots, q$ satisfying*

$$|f_j(x) - r_j| \leq \alpha|x - r_j|,$$

for all x where $0 \leq \alpha < 1$. Then for any initial value x_1, the solution of the difference equation

$$x_n = \max\{f_1(x_{n-1}), \ldots, f_q(x_{n-1})\}, \tag{8}$$

converges to $\max_j r_j$.

Example 2.9. In analogy with Example 2.6, the solution of the difference equation

$$x_n = \max\left\{A_1 x_{n-1}^{\alpha_1}, \ldots, A_q x_{n-1}^{\alpha_q}\right\}, \tag{9}$$

where $A_j > 0, -1 < \alpha_j < 1$ for $j = 1, \ldots, q$ is convergent to

$$\max_{1 \leq j \leq q} A_j^{1/(1-\alpha_j)}.$$

for any initial value x_1, according to Corollary 2.8.

Example 2.10. Theorem 2.3 establishes convergence of the difference equation

$$x_n = \max\{a_1 + \sin b_1 x_{n-1}, c_1 + \cos d_1 x_{n-1}, a_2 + \sin b_2 x_{n-2}, c_2 + \cos d_2 x_{n-2}\},$$

where we assume $\alpha = \max\{|b_1|, |b_2|, |d_1|, |d_2|\} < 1$. Define the functions $f_{i1}(x) = a_i + \sin b_i x$ and $f_{i2}(x) = c_i + \cos d_i x$. Under the assumptions, for $1 \leq i, j \leq 2$, $|f'_{ij}(x)| \leq \alpha < 1$ for all x, and by the mean value theorem, each f_{ij} is globally contractive to a unique fixed point r_{ij}. (In particular, each r_{ij} is the unique solution of the equation $x = f_{ij}(x)$.) Thus, Theorem 2.3 can be applied with $p = q_1 = q_2 = 2$ to conclude that for any set of initial values $\{x_1, x_2\}$,

$$\lim_{n\to\infty} x_n = \max_{1 \leq i \leq 2, 1 \leq j \leq 2} r_{ij}.$$

3. Local convergence

Definition 3.1. *We call the constant solution $x_n = r$ of a difference equation locally attractive if for some p-dimensional open neighbourhood of initial values $(x_1, \ldots, x_p) = (r, \ldots, r)$, the solution converges to the constant solution r.*

This definition concerns local convergence, for cases when nearby initial values, but perhaps not all initial values, are attracted to a given constant solution. In the context of max-type equations, in order to make conclusions about local convergence, an extra hypothesis that is not strictly local needs to be added to control the contractivity between the individual fixed points, as shown in the next theorem.

THEOREM 3.2. *Consider p nonnegative integers* $q_1, \ldots, q_p$ *and let* $0 \leq \alpha < 1$. *Assume for each i, j satisfying* $1 \leq i \leq p, 1 \leq j \leq q_i$ *there exists a continuously differentiable function* $f_{ij} : R \to R$ *and a real number* r_{ij} *satisfying* $f_{ij}(r_{ij}) = r_{ij}$. *Let* i_m, j_m *be integers satisfying* $r_{i_m j_m} = \max_{i,j} r_{ij}$. *Assume that for each* $i, j, |f'_{ij}(x)| \leq \alpha$ *for* $r_{ij} \leq x \leq r_{i_m j_m}$. *Then the constant solution* $x_n = r_{i_m j_m}$ *of the difference equation*

$$x_n = \max_{1 \leq i \leq p, 1 \leq j \leq q_i} \{f_{ij}(x_{n-i})\}, \tag{10}$$

is locally attractive.

Proof. Choose $\epsilon > 0$ such that for each i, j, $|f'_{ij}(x)| \leq \alpha_1 \equiv (\alpha + 1)/2 < 1$ for $r_{ij} - \epsilon < x < r_{i_m j_m} + \epsilon$. For each i, j and $r_{ij} - \epsilon < x < r_{i_m j_m} + \epsilon$, the mean value theorem implies $|f_{ij}(x) - r_{ij}| \leq \alpha_1 |x - r_{ij}|$. Define the open set $U = \{(x_1, \ldots, x_p) : |x_i - r_{i_m j_m}| < \epsilon, 1 \leq i \leq p\}$.

The remainder of the proof closely parallels the proof of Theorem 2.3. Choose $(x_1, \ldots, x_p)$ from U and for each $n > p$, choose i', j' such that $x_n = \max_{i,j} \{f_{ij}(x_{n-i})\} = f_{i'j'}(x_{n-i'})$. Apply Lemma 2.2 with $u_1 = x_{n-i_m}, y_1 = f_{i_m j_m}(x_{n-i_m}), u_2 = x_{n-i'}, y_2 = f_{i'j'}(x_{n-i'}), s_1 = r_{i_m j_m}$ and $s_2 = r_{i'j'}$. Lemma 2.2 implies that

$$|x_n - r_{i_m j_m}| = |\max_{i,j} \{f_{ij}(x_{n-i})\} - r_{i_m j_m}| \leq \alpha_1 |z - r_{i_m j_m}|,$$

where $z = x_{n-i_m}$ or $x_{n-i'}$. This implies by induction that (a) x_n belongs to U and (b) we can apply Lemma 2.1 to conclude that $\lim_{n \to \infty} x_n = r_{i_m j_m}$. □

The $q_i \equiv 1$ special case is the local version of Corollary 2.4.

COROLLARY 3.3. *Assume that the continuously differentiable functions* $f_i : R \to R$ *and real numbers* r_i *for* $i = 1, \ldots, p$ *satisfy* $f_i(r_i) = r_i$. *Let* i_m *be an integer satisfying* $r_{i_m} = \max_{1 \leq i \leq p} r_i$, *and assume that there exists* $0 \leq \alpha < 1$ *such that for* $1 \leq i \leq p$, $|f'_i(x)| \leq \alpha$ *for* $r_i \leq x \leq r_{i_m}$. *Then the constant solution* $x_n = r_{i_m}$ *of the difference equation*

$$x_n = \max\{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\}, \tag{11}$$

is locally attractive.

Example 3.4. Define

$$x_n = \max\{x_{n-1} e^{a_1(1 - x_{n-1}/c_1)}, \ldots, x_{n-p} e^{a_p(1 - x_{n-p}/c_p)}\}, \tag{12}$$

where each $f_i(x) = x e^{a_i(1 - x/c_i)}$ in (11) is a Ricker map [10] with growth parameter $a_i \geq 0$ and carrying capacity $c_i \geq 0$. Corollary 2.4 does not apply in this range since $f'_i(0) = e^{a_i} \geq 1$ and so f_i is not contractive.

If $0 < a_i < 2$, it is easily checked that c_i is a stable fixed point for the individual map f_i. In fact, the derivative of $f_i(x) = x\mathrm{e}^{a_i(1-x/c_i)}$ is $f_i'(x) = (1 - a_i x/c_i)\mathrm{e}^{a_i(1-x/c_i)}$, and so $|f_i'(c_i)| = |1 - a_i| < 1$. Furthermore, the second derivative shows that $f_i'(x)$ is decreasing on the interval $[c_i, 2c_i/a_i)$ from $f_i'(c_i) = 1 - a_i$ to $f_i'(2c_i/a_i) = -\mathrm{e}^{a_i-2}$, and increasing on the interval $(2c_i/a_i, \infty)$ from $f_i'(2c_i/a_i) = -\mathrm{e}^{a_i-2}$ to 0. We conclude that $|f_i'(x)| \leq \max\{|1 - a_i|, \mathrm{e}^{a_i-2}\} < 1$ for $c_i \leq x$. Using this fact, we can verify the main hypothesis of Corollary 3.3, that for each i, $|f_i'(x)| = |(1 - a_i x/c_i)\mathrm{e}^{a_i(1-x/c_i)}| \leq \alpha \equiv \max_i\{|1 - a_i|, \mathrm{e}^{a_i-2}\} < 1$ for $c_i < x < c_{i_m}$. Therefore, the constant solution $\{c_{i_m}, c_{i_m}, \ldots\}$ is locally attractive for the max-type equation (12), where $c_{i_m} = \max\{c_i\}$ is the maximum of the carrying capacities of the p individual Ricker maps.

Interestingly, the result is independent of the relative values of the a_i, as long as they lie in the range $(0, 2)$. This solution is not globally attractive, since for example the zero solution does not converge to it. However, since the zero solution is unstable, the solution of (12) converges to c_{i_m} for initial data near zero, and in fact for almost every positive initial condition.

The $p = 1$ special case of Theorem 3.2 is the local version of Corollary 2.8.

Corollary 3.5. *Let q be a positive integer and consider real numbers r_j and continuously differentiable functions $f_j : R \to R$ for $j = 1, \ldots, q$ satisfying $f_j(r_j) = r_j$. Let m be an integer satisfying $r_{j_m} = \max_{1 \leq j \leq q} r_j$, and assume that there exists $0 \leq \alpha < 1$ such that for $1 \leq j \leq q$, $|f_j'(x)| \leq \alpha$ for $r_j \leq x \leq r_{j_m}$. Then the constant solution $x_n = r_{j_m}$ of the difference equation*

$$x_n = \max\{f_1(x_{n-1}), \ldots, f_q(x_{n-1})\}, \tag{13}$$

is locally attractive.

Example 3.6. Consider the difference equation

$$x_n = \max\{(x_{n-1} - a_1)^2, (x_{n-1} - a_2)^2, \ldots, (x_{n-1} - a_q)^2\}, \tag{14}$$

where $-(1/4) < a_j < 3/4$ for $j = 1, \ldots, q$. The fixed point $r_j = a_j + (1/2) - \sqrt{a_j + (1/4)}$ is an attracting fixed point of $f_j(x) = (x - a_j)^2$. Note that each fixed point lies in the interval $[0, 1/4)$.

In addition, note that for each j and for x between $x = r_j$ and $x = 1/4$, $f_j'(x)$ is increasing from $f_j'(r_j) = 1 - \sqrt{4a_j + 1}$ to $f_j'(1/4) = 2(1/4 - a_j)$, so that $|f_j'(x)| \leq \max\{|1 - \sqrt{4a_j + 1}|, |2(1/4 - a_j)|\} < 1$, satisfying the main hypothesis of Corollary 3.5. Therefore, the constant solution $x_n = r_{j_m}$, the maximum of the q individually attracting fixed points of the f_j, is locally attractive for the max-type equation (14).

Remark 1. Analogues of the convergence Theorems 2.3 and 3.2 also hold for min-type difference equations, by applying the max versions to $-f_i(x)$. For example, the local version for min-type equations takes the following form.

Theorem 3.7. *Consider p nonnegative integers $q_1, \ldots, q_p$, and let $0 \leq \alpha < 1$. Assume for each i, j satisfying $1 \leq i \leq p, 1 \leq j \leq q_i$ there exists a continuously differentiable function $f_{ij} : R \to R$ and a real number r_{ij} satisfying $f_{ij}(r_{ij}) = r_{ij}$. Let i_m, j_m be integers* *satisfying* $r_{i_m j_m} = \min_{i,j} r_{ij}$. *Assume that for each* i,j, $|f'_{ij}(x)| \le \alpha$ *for* $r_{i_m j_m} \le x \le r_{ij}$. *Then the constant solution* $x_n = r_{i_m j_m}$ *of the difference equation*

$$x_n = \min_{1\le i\le p, 1\le j\le q_i} \{f_{ij}(x_{n-i})\} \tag{15}$$

is locally attractive.

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References

[1] K. Berenhaut, J. Foley, and S. Stevic, *Boundedness character of positive solution of a max difference equation*, J. Difference Equ. Appl. 12 (2006), pp. 1183–1189.
[2] W.J. Bride, E.A. Grove, C.M. Kent, and G. Ladas, *Eventually periodic solutions of* $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$, Comm. Appl. Nonlinear Anal. 6 (1999), pp. 31–34.
[3] Y. Chen, *Eventually periodicity of* $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ *with periodic coefficients*, J. Difference Equ. Appl. 11 (2005), pp. 1289–1294.
[4] C. Cinar, S. Stevic, and I. Yalcinkaya, *On positive solutions of a reciprocal difference equation with minimum*, J. Appl. Math. Comput. 17 (2005), pp. 307–314.
[5] J. Feuer, *On the eventual periodicity of* $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ *with a period-four parameter*, J. Difference Equ. Appl. 12 (2006), pp. 467–486.
[6] E.A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, Chapman & Hall/CRC Press, 2005.
[7] E.A. Grove, C. Kent, G. Ladas, and M. Radin, *On* $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$ *with a period 3 parameter*, Fields Inst. Commun. 29 (2001), pp. 161–180.
[8] C.M. Kent and M.A. Radin, *On the boundedness nature of positive solutions of the difference equation* $x_{n+1} = \max\{A_n/x_n, B_n/x_{n-1}\}$ *with periodic parameters*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithm (2003), pp. 11–15.
[9] G. Ladas, *On the recursive sequence* $x_{n+1} = \max\{A_0/x_n, \ldots, A_k/x_{n-k}\}$, J. Difference Equ. Appl. 2 (1996), pp. 339–341.
[10] W.E. Ricker, *Stock and recruitment*, J. Fish. Res. Board Can. 11 (1954), pp. 559–623.
[11] S. Stevic, *On the recursive sequence* $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$, Appl. Math. Lett. 21 (2008), pp. 791–796.
[12] S. Stevic, *Global stability of a difference equation with maximum*, Appl. Math. Comput. 210 (2009), pp. 525–529.
[13] F. Sun, *On the asymptotic behavior of a difference equation with maximum*, Discrete Dyn. Nat. Soc. (2008), 243291.
[14] I. Szalkai, *On the periodicity of the sequence* $x_{n+1} = \max\{A_0/x_n, \ldots, A_k/x_{n-k}\}$, J. Difference Equ. Appl. 5 (1999), pp. 25–29.