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Convergence of rank-type equations

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ABSTRACT

Convergence results are presented for rank-type difference equations, whose evolution rule is defined at each step as the kth largest of p univariate difference equations. If the univariate equations are individually contractive, then the equation converges to a fixed point equal to the kth largest of the individual fixed points of the univariate equations. Examples are max-type equations for $k = 1$, and the median of an odd number p of equations, for $k = (p + 1)/2$. In the non-hyperbolic case, conjectures are stated about the eventual periodicity of the equations, generalizing long-standing conjectures of G. Ladas.

1. Introduction

For a set of p real numbers $\{r_1,\ldots,r_p\}$, denote the kth-largest element of the set by k-rank$\{r_1,\ldots,r_p\}$. Thus 2-rank$\{6,2,5,3\} = 5$, and 1-rank is synonymous with max.

Let f_i: $R \to R$ for $i = 1,\ldots,p$ be real-valued functions. Consider the difference equation

$$x_n = k - \operatorname{rank}\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\} \tag{1}$$

for initial data $x_1, \ldots, x_p$. We will call such an equation a *rank-type* difference equation. If the f_i are continuous, then x_n is a continuous function of $x_{n-1}, \ldots, x_{n-p}$. Special cases of rank-type equations include

$$x_n = \max\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\}, \tag{2}$$
$$x_n = \min\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\} \tag{3}$$

and

$$x_n = \operatorname{median}\{f_1(x_{n-1}), f_2(x_{n-2}), \ldots, f_p(x_{n-p})\} \tag{4}$$

in the case where p is odd.

Max-type equations, corresponding to the special case $k = 1$ in difference Eq. (1), have been extensively studied [1–9,14,13,17–21]. The purpose of this article is to note that, perhaps surprisingly, many of the properties of max-type equations are shared more generally by rank-type equations for $k > 1$.

Definition. The function f is called *contractive* if there exists $0 \leqslant \alpha < 1$ and a real number r such that $|f(x) - r| \leqslant \alpha|x - r|$ for all x.

Definition. The solution $\{x_n\}_{n=1}^{\infty}$ of a difference equation is called *globally convergent* if there exists r such that for every set of initial values, $\lim_{n\to\infty} x_n = r$. In this case, the equilibrium r is called *globally attractive*.

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In the next section we show that if the f_i are contractive with fixed points r_i, then the difference Eq. (1) is globally convergent for any set $\{x_1,\ldots,x_p\}$ of initial values, converging in the limit to the fixed point k-rank $\{r_1,\ldots,r_p\}$.

This result is a generalization of the convergence theorem for max-type equations, the $k = 1$ case [11]. If p is odd and $k = (p+1)/2$, then the convergence result corresponds to replacing max by median. The corresponding statement for mean is false; see Example 2.6.

Theorem 2.3 below is the main global convergence result, proved in a context slightly more general than (1). The techniques used to prove Theorem 2.3 can also be applied to prove a local convergence version, Theorem 3.1. In the final section, we relax the hyperbolicity restriction and state some conjectures, generalizing well-known conjectures of Ladas [9] and Grove and Ladas [7] on max-type equations.

2. Global convergence

In this section, we prove the main convergence result for rank-type maps after two preliminary lemmas.

Lemma 2.1. *Let p be a positive integer, r and $0 \leqslant \alpha < 1$ real numbers, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Assume that for each n there exists i, possibly depending on n, $1 \leqslant i \leqslant p$, such that $|x_n - r| \leqslant \alpha |x_{n-i} - r|$. Then $\lim_{n\to\infty} x_n = r$.*

The proof of Lemma 2.1 can be found in [12] as well as [11]. The next lemma generalizes Lemma 2.2 of [11].

Lemma 2.2. *Let $u_1, u_2, y_1 \leqslant y_2$, and $s_2 \leqslant r \leqslant s_1$ be real numbers, and assume $|y_i - s_i| \leqslant \alpha |u_i - s_i|$ for $i = 1, 2$ for some $0 \leqslant \alpha < 1$. Then*

(i) *$|y_2 - r| \leqslant \alpha |u_j - r|$ for either $j = 1$ or $j = 2$, and*
(ii) *$|y_1 - r| \leqslant \alpha |u_j - r|$ for either $j = 1$ or $j = 2$.*

Proof. We give the proof of (i). The proof of (ii) follows by applying (i) to $-u_1, -u_2, -y_2, -y_1, -s_1, -r, -s_2$.

The proof of (i) is divided into four cases.

Case 1: $y_2 \leqslant r$, $u_1 \geqslant s_1$. In this case,

$$|y_2 - r| = r - y_2 \leqslant s_1 - y_2 \leqslant s_1 - y_1 = |s_1 - y_1| \leqslant \alpha |s_1 - u_1| = \alpha(u_1 - s_1) \leqslant \alpha(u_1 - r) = \alpha |u_1 - r|.$$

Case 2: $y_2 \leqslant r$, $u_1 \leqslant s_1$.

$$\begin{aligned}|y_2 - r| &= r - y_2 \leqslant r - y_1 = r - s_1 + s_1 - y_1 = r - s_1 + |y_1 - s_1| \leqslant r - s_1 + \alpha |u_1 - s_1| = r - s_1 + \alpha(s_1 - u_1)\\ &\leqslant r - s_1 + \alpha(s_1 - u_1) + (1-\alpha)(s_1 - r) = \alpha(r - u_1) \leqslant \alpha |r - u_1|.\end{aligned}$$

Case 3: $y_2 \geqslant r$, $u_2 \geqslant r$.

$$\begin{aligned}|y_2 - r| &= y_2 - r = y_2 - s_2 + s_2 - r = |y_2 - s_2| + s_2 - r \leqslant \alpha |u_2 - s_2| + s_2 - r = \alpha(u_2 - s_2) + s_2 - r\\ &\leqslant \alpha(u_2 - s_2) + s_2 - r + (1-\alpha)(r - s_2) = \alpha(u_2 - r) = \alpha |u_2 - r|.\end{aligned}$$

Case 4: $y_2 \geqslant r$, $u_2 \leqslant r$.
If $u_2 \geqslant s_2$, then

$$r - s_2 \leqslant y_2 - s_2 = |y_2 - s_2| \leqslant \alpha |u_2 - s_2| = \alpha(u_2 - s_2) \leqslant \alpha(r - s_2),$$

a contradiction. So in addition, we may assume $u_2 < s_2 \leqslant r$. Then

$$|y_2 - r| = y_2 - r \leqslant y_2 - s_2 = |y_2 - s_2| \leqslant \alpha |u_2 - s_2| = \alpha(s_2 - u_2) \leqslant \alpha(r - u_2) = \alpha |u_2 - r|,$$

which completes the proof. □

Theorem 2.3. *Consider p nonnegative integers $q_1, \ldots, q_p$, and let $0 \leqslant \alpha < 1$. Assume for each i, j satisfying $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q_i$ there exists a function f_{ij}: $R \to R$ and a real number r_{ij} satisfying*

$$|f_{ij}(x) - r_{ij}| \leqslant \alpha |x - r_{ij}|$$

for all x. Then for any k and for any set $\{x_1,\ldots,x_p\}$ of initial values, the solution of the difference equation

$$x_n = k\text{-}rank_{1\leqslant i\leqslant p, 1\leqslant j\leqslant q_i}\{f_{ij}(x_{n-i})\} \tag{5}$$

converges to k-$rank_{1\leqslant i\leqslant p, 1\leqslant j\leqslant q_i} r_{ij}$, the kth-largest of the r_{ij}.

Proof. There are $q \equiv \sum_{i=1}^{p} q_j$ functions f_{ij}, each with fixed point r_{ij}. Rank the q fixed points as follows:

$$r_{i_1 j_1} \geqslant r_{i_2 j_2} \geqslant \cdots \geqslant r_{i_q j_q}.$$

We need to prove that the sequence $x_n = k\text{-rank}\{f_{ij}(x_{n-i})\}$ converges to $r \equiv r_{i_kj_k} = k\text{-rank}\ r_{ij}$.

For a fixed n, define i_m, j_m so that $x_n \equiv k\text{-rank}_{i,j}\{f_{ij}(x_{n-i})\} = f_{i_mj_m}(x_{n-i_m})$. To apply Lemma 2.1, we will find x_{n-i} satisfying $|x_n - r| \leqslant \alpha|x_{n-i} - r|$ where $1 \leqslant i \leqslant p$.

If $m = k$, then $|x_n - r| \leqslant \alpha|x_{n-i_k} - r|$, as required. If $m < k$, then there exists an integer $h \in \{k, k+1, \ldots, q\}$ such that $f_{i_hj_h}(x_{n-i_h}) \geqslant f_{i_mj_m}(x_{n-i_m})$. Thus $m < k \leqslant h$, or in other words, $r_{i_mj_m} \geqslant r_{i_kj_k} \geqslant r_{i_hj_h}$. Now we can apply Lemma 2.2 with $s_2 = r_{i_hj_h} \leqslant r = r_{i_kj_k} \leqslant s_1 = r_{i_mj_m}$, $y_1 = f_{i_mj_m}(x_{n-i_m}) \leqslant y_2 = f_{i_hj_h}(x_{n-i_h})$, and $u_1 = x_{n-i_m}$, $u_2 = x_{n-i_h}$. The result of part (ii) of the lemma is that

$$|x_n - r| = |k\text{-rank}_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q_i}\{f_{ij}(x_{n-i})\} - r| \leqslant \alpha|z - r|, \tag{6}$$

where $z = x_{n-i_m}$ or x_{n-i_h}, as required. Finally, if $m > k$, there exists $h \in \{1, 2, \ldots, k\}$ such that $f_{i_hj_h}(x_{n-i_h}) \leqslant f_{i_mj_m}(x_{n-i_m})$. Thus $h \leqslant k < m$, or in other words, $r_{i_hj_h} \geqslant r_{i_kj_k} \geqslant r_{i_mj_m}$. Part (i) of Lemma 2.2 with $s_2 = r_{i_mj_m} \leqslant r = r_{i_kj_k} \leqslant s_1 = r_{i_hj_h}$, $y_1 = f_{i_hj_h}(x_{n-i_h}) \leqslant y_2 = f_{i_mj_m}(x_{n-i_m})$, and $u_1 = x_{n-i_h}$, $u_2 = x_{n-i_m}$ yields (6) as before.

This satisfies the hypotheses of Lemma 2.1, so

$$\lim_{n\to\infty} x_n = r = k\text{-rank}\ \{r_{ij}\}. \qquad \square$$

Setting all $q_i = 1$ in Theorem 2.3 covers the special case referred to as Eq. (1) in the introduction.

Corollary 2.4. *Let $r_1, \ldots, r_p$ be real numbers and assume f_i: $R \to R$ for $i = 1, \ldots, p$ satisfy $|f_i(x) - r_i| \leqslant \alpha|x - r_i|$ for all x, where $0 \leqslant \alpha < 1$. Then for any set $\{x_1, \ldots, x_p\}$ of initial values, the solution of difference equation*

$$x_n = k\text{-rank}\ \{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\} \tag{7}$$

converges to k-rank $\{r_1, \ldots, r_p\}$ as $n \to \infty$.

Example 2.5. As an application of Corollary 2.4, consider the difference equation

$$x_n = k\text{-rank}\left\{\frac{1}{a_1 + b_1x_{n-1}^2}, \ldots, \frac{1}{a_p + b_px_{n-p}^2}\right\}, \tag{8}$$

where $0 < a_i$, $0 \leqslant b_i < (4a_i/3)^3$ for $i = 1, \ldots, p$. Under these conditions, for each i, the first derivative of $f_i(x) = 1/(a_i + b_ix^2)$ is smaller than 1 in absolute value for all x. By the Mean Value Theorem, the hypotheses of Corollary 2.4 hold where r_i denotes the real root of the equation $b_ix^3 + a_ix = 1$. Therefore all solutions of (8) must converge to k-rank $\{r_1, \ldots, r_p\}$.

A particular case of (8) for $k = 2$ is the difference equation

$$x_n = \text{median}\left\{\frac{1}{1.2 + 0.7x_{n-1}^2}, \frac{1}{1 + x_{n-2}^2}, \frac{1}{1.1 + 0.9x_{n-3}^2}\right\}. \tag{9}$$

The fixed points of

$$f_1(x) = \frac{1}{1.2 + 0.7x^2}, \quad f_2(x) = \frac{1}{1 + x^2}, \quad f_3(x) = \frac{1}{1.1 + 0.9x^2}$$

are approximately $r_1 = 0.6632$, $r_2 = 0.6823$, and exactly $r_3 = 2/3$, respectively. Corollary 2.4 implies that all solutions of (9) converge to $r_3 = 2/3$, the median of the three fixed points.

Remark. Corollary 2.4 implies that if p is odd, then the median difference Eq. (4) converges to the median of the individual fixed points of $f_1, \ldots, f_p$. However, if p is even, and we follow the usual convention that the median refers to the average of the two middle ranked entries, this statement fails to hold, as shown in the next example.

Example 2.6. If $a_1 + a_2 < 2$ for positive numbers a_1, a_2, the equation

$$x_n = \text{median}\{a_1x_{n-1} + b_1, a_2x_{n-2} + b_2\} \tag{10}$$

is equivalent to

$$x_n = \text{mean}\{a_1x_{n-1} + b_1, a_2x_{n-2} + b_2\}$$

and converges to the fixed point

$$\lim_{n\to\infty} x_n = (b_1 + b_2)/(2 - a_1 - a_2).$$

This disagrees in general with the mean of the fixed points of $f_1(x) = a_1x + b_1$ and $f_2(x) = a_2x + b_2$, which is $b_1/2(1 - a_1) + b_2/2(1 - a_2)$.

Example 2.7. Consider the difference equation

$$x_n = k\text{-rank}\{A_1x_{n-1}^{\alpha_1}, \ldots, A_px_{n-p}^{\alpha_p}\}, \tag{11}$$

where $A_i > 0$, $-1 < \alpha_i < 1$ for $i = 1, \ldots, p$, and $x_1, \ldots, x_p$ are initial values. Set $y_n = \log x_n$. In these coordinates, the ith equation is $y_n = \alpha_i y_{n-i} + \log A_i$, and due to monotonicity of the logarithm, (11) is replaced with

$$y_n = k\text{-rank}\{\alpha_1 y_{n-1} + \log A_1, \ldots, \alpha_p y_{n-p} + \log A_p\}.$$

Corollary 2.4 shows that for any set of positive initial values $x_1, \ldots, x_p$, the y_n sequence converges to k-rank $\{\log A_i/(1-\alpha_i)\}$, so that

$$\lim_{n\to\infty} x_n = k\text{-rank}_{1\leqslant i\leqslant p} A_i^{\frac{1}{1-\alpha_i}}.$$

This proves asymptotic convergence of (11) for $-1 < \alpha_i < 1$, $A_i > 0$, and for all positive initial conditions. The max-type version of the problem, corresponding to $k = 1$, was previously treated in [11,14–17].

3. Local convergence

Definition. The constant solution $x_n = r$ of a difference equation will be called *locally attractive* if for some p-dimensional open neighborhood of initial values $(x_1,\ldots,x_p) = (r,\ldots,r)$, the solution converges to the constant solution r.

This definition concerns local convergence, for cases when nearby initial values are attracted to a given constant solution. In the context of rank-type equations, in order to make conclusions about local convergence, an extra hypothesis that is not strictly local needs to be added to control the contractivity between the individual fixed points, as shown in the next theorem.

Theorem 3.1. *Consider p nonnegative integers $q_1, \ldots, q_p$, and let $0 \leqslant \alpha < 1$. Assume for each i, j satisfying $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q_i$ there exists a continuously differentiable function f_{ij}: $R \to R$ and a real number r_{ij} satisfying $f_{ij}(r_{ij}) = r_{ij}$. Let i_k, j_k be integers satisfying $r_{i_kj_k} = k\text{-rank}_{i,j} r_{ij}$. Assume that for each i, j, $|f'_{ij}(x)| \leqslant \alpha$ for x between r_{ij} and $r_{i_kj_k}$. Then the constant solution $x_n = r_{i_kj_k}$ of the rank-type difference equation*

$$x_n = k\text{-rank}_{1\leqslant i\leqslant p, 1\leqslant j\leqslant q_i}\{f_{ij}(x_{n-i})\} \tag{12}$$

is locally attractive.

Proof. Choose $\epsilon > 0$ such that for each i, j, $|f'_{ij}(x)| \leqslant \alpha_1 \equiv (\alpha+1)/2 < 1$ for $r_{ij} - \epsilon < x < r_{i_kj_k} + \epsilon$. For each i, j and $r_{ij} - \epsilon < x < r_{i_kj_k} + \epsilon$, the Mean Value Theorem implies $|f_{ij}(x) - r_{ij}| \leqslant \alpha_1|x - r_{ij}|$. Define the open set $U = \{(x_1,\ldots,x_p) : |x_i - r_{i_kj_k}| < \epsilon, 1 \leqslant i \leqslant p\}$.

The remainder of the proof is similar to the proof of Theorem 2.3. Choose $(x_1,\ldots,x_p)$ from U, and for each $n > p$, choose i', j' such that $x_n = \max_{i,j}\{f_{ij}(x_{n-i})\} = f_{i'j'}(x_{n-i'})$. Apply Lemma 2.2 with $u_1 = x_{n-i_k},\ y_1 = f_{i_kj_k}(x_{n-i_k}),\ u_2 = x_{n-i'},\ y_2 = f_{i'j'}(x_{n-i'}),\ s_1 = r_{i_kj_k}$, and $s_2 = r_{i'j'}$. Lemma 2.2 implies that

$$|x_n - r_{i_kj_k}| = |\max_{i,j}\{f_{ij}(x_{n-i})\} - r_{i_kj_k}| \leqslant \alpha_1|z - r_{i_mj_m}|,$$

where $z = x_{n-i_k}$ or $x_{n-i'}$. This implies that (a) x_n belongs to U and (b) we can apply Lemma 2.1 to conclude that $\lim_{n\to\infty} x_n = r_{i_kj_k}$. $\square$

The $q_i \equiv 1$ special case is the local version of Corollary 2.4.

Corollary 3.2. *Assume that the continuously differentiable functions f_i: $R \to R$ and real numbers r_i for $i = 1, \ldots, p$ satisfy $f_i(r_i) = r_i$. Let i_k be an integer satisfying $r_{i_k} = k\text{-rank}_{1\leqslant i\leqslant p} r_i$, and assume that there exists $0 \leqslant \alpha < 1$ such that for $1 \leqslant i \leqslant p$, $|f'_i(x)| \leqslant \alpha$ for x between r_i and r_{i_k}. Then the constant solution $x_n = r_{i_k}$ of the difference equation*

$$x_n = k\text{-rank}\{f_1(x_{n-1}), \ldots, f_p(x_{n-p})\} \tag{13}$$

is locally attractive.

We revisit two examples of max-type equations from [11], and discuss them in the more general context of Corollary 3.2.

Example 3.3. As a first example, consider the rank-type equation involving Ricker maps [10]

$$x_n = k\text{-rank}\{x_{n-1}e^{a_1(1-x_{n-1}/c_1)}, \ldots, x_{n-p}e^{a_p(1-x_{n-p}/c_p)}\}, \tag{14}$$

where each map $f_i(x) = xe^{a_i(1-x/c_i)}$ in (13) has growth parameter $a_i \geqslant 0$ and carrying capacity $c_i \geqslant 0$. Since $f'_i(0) = e^{a_i} \geqslant 1$, f_i is not contractive, and the hypotheses of Corollary 2.4 are not satisfied.

However, note that if $0 < a_i < 2$, then c_i is a stable fixed point for f_i, since the derivative of $f_i(x) = xe^{a_i(1-x/c_i)}$ is $f'_i(x) = (1 - a_ix/c_i)e^{a_i(1-x/c_i)}$, and $|f'_i(c_i)| = |1 - a_i| < 1$. In addition, the second derivative shows that $f'_i(x)$ is decreasing on the interval $[c_i, 2c_i/a_i)$ from $f'_i(c_i) = 1 - a_i$ to $f'_i(2c_i/a_i) = -e^{a_i-2}$, and increasing on the interval $(2c_i/a_i, \infty)$ from $f'_i(2c_i/a_i) = -e^{a_i-2}$ to 0. It follows immediately that $|f'_i(x)| \leqslant \max\{|1 - a_i|, e^{a_i-2}\} < 1$ for $c_i \leqslant x$. Now the main hypothesis of Corollary 3.2, that for each $i, |f_i'(x)| = |(1-a_i x/c_i)e^{a_i(1-x/c_i)}| \leqslant \alpha \equiv \max_i\{|1-a_i|, e^{a_i-2}\} < 1$ for x between c_i and c_{i_k}, is verified. Therefore the constant solution $\{c_{i_k}, c_{i_k}, \ldots\}$ is locally attractive for the rank-type Eq. (14), where $c_{i_k} = k\text{-rank}\{c_i\}$ is the kth-largest of the carrying capacities of the p individual Ricker maps.

Example 3.4. Assume $-1/4 < a_i < 3/4$ for $1 \leqslant i \leqslant p$. Then the fixed point $r_i = a_i + \frac{1}{2} - \sqrt{a_i + \frac{1}{4}}$ of $f_i(x) = (x - a_i)^2$ is an attracting fixed point. Note that each fixed point lies in the interval $[0, \frac{1}{4})$.

We can apply Corollary 3.2 to the difference equation

$$x_n = k\text{-rank}\Big\{(x_{n-1} - a_1)^2, (x_{n-2} - a_2)^2, \ldots, (x_{n-p} - a_p)^2\Big\}. \tag{15}$$

Note that for each j and for x between $x = r_i$ and $x = \frac{1}{4}, f_i'(x)$ is increasing from $f_i'(r_i) = 1 - \sqrt{4a_i + 1}$ to $f_i'(\frac{1}{4}) = 2(\frac{1}{4} - a_i)$, so that $|f_i'(x)| \leqslant \max\{|1 - \sqrt{4a_i + 1}|, |2(\frac{1}{4} - a_i)|\} < 1$, satisfying the main hypothesis of Corollary 3.2. It follows that the constant solution $x_n = r_{i_k}$, the kth-largest of the p individually attracting fixed points of the f_i, is locally attractive for the rank-type Eq. (15).

4. Non-hyperbolic case

Far less is known in the non-hyperbolic case, where the restriction that $\alpha < 1$ is relaxed. Because of the lack of hyperbolicity, convergence cannot be expected for general initial conditions. In many cases the solution becomes periodic for sufficiently large n. A particularly rich case concerns the difference equation

$$x_n = k\text{-rank}\{-x_{n-1} + b_1, \ldots, -x_{n-p} + b_p\}. \tag{16}$$

Remark. First define i_k to be the index of the kth-largest b_i, namely $b_{i_k} = k\text{-rank}\ \{b_1, \ldots, b_p\}$. Then it is straightforward to check that (16) has a fixed-point solution $x_1 = x_2 = \cdots = \frac{1}{2} b_{i_k}$.

With more assumptions, many more periodic solutions can be generated. We say that a solution has *prime period s* if it is periodic with period s and with no lower period.

Proposition 4.1. *Assume that the $b_1, \ldots, b_p$ in (16) are ordered as*

$$b_{i_1} \geqslant \cdots \geqslant b_{i_{k-1}} > b_{i_k} > b_{i_{k+1}} \geqslant \cdots \geqslant b_{i_p}.$$

That is, assume that b_{i_k}, the kth-largest b_i, is not repeated in the list. Then there are uncountably many solutions of (16) with prime period $2i_k$.

The solutions can be described as follows. Define

$$\beta = \frac{1}{2}\min\{b_{i_{k-1}} - b_{i_k}, b_{i_k} - b_{i_{k+1}}\}.$$

(If $k = 1$, set $\beta = \frac{1}{2}(b_{i_k} - b_{i_{k+1}})$; if $k = p$, set $\beta = \frac{1}{2}(b_{i_{k-1}} - b_{i_k})$). Define the p initial conditions $x_1, \ldots, x_p$ to be any p consecutive elements of the sequence

$$x_1, \ldots, x_{2i_k}, x_1, \ldots, x_{2i_k}, \ldots$$

where

$$x_1 = \frac{b_{i_k}}{2} + \beta_1,$$

$$\vdots$$

$$x_{i_k} = \frac{b_{i_k}}{2} + \beta_{i_k},$$

$$x_{i_k+1} = \frac{b_{i_k}}{2} - \beta_1,$$

$$\vdots$$

$$x_{2i_k} = \frac{b_{i_k}}{2} - \beta_{i_k}$$

and such that the β_i satisfy $|\beta_i| < \beta$. For each n,

$$x_n = k\text{-rank}\Big\{-\frac{b_{i_k}}{2} - \beta_{j_1} + b_1, \ldots, -\frac{b_{i_k}}{2} - \beta_{j_p} + b_p\Big\}.$$

Since $b_{i_{k-1}} - 2\beta \geqslant b_{i_k} \geqslant b_{i_{k+1}} + 2\beta$, we have

$$-\frac{b_{i_k}}{2} + b_{i_{k-1}} - 2\beta \geqslant -\frac{b_{i_k}}{2} + b_{i_k} \geqslant -\frac{b_{i_k}}{2} + b_{i_{k+1}} + 2\beta,$$

so that the kth largest of the set is $-x_{n-i_k} + b_{i_k}$. This verifies that each x_n defined by the difference Eq. (16) follows the same pattern, and satisfies $x_n = x_{n-2i_k}$. If the $\beta_1, \ldots, \beta_k$ are chosen all distinct, then the solution is not periodic of any lower period, so its prime period is $2i_k$.

Remark. In addition to the solutions of prime period $2i_k$, there are solutions of prime period $2i_k/d$ for any odd natural number d dividing evenly into i_k. They are special cases of the above solutions obtained by setting $\beta_{s+1} = -\beta_1$, $\beta_{s+2} = -\beta_2, \ldots, \beta_{2s} = -\beta_s$, $\beta_{2s+1} = \beta_1, \ldots$, where $s = i_k/d$.

Note that difference equation (16) is the additive version of the multiplicative difference equation

$$x_n = k\text{-rank}\left\{\frac{A_1}{x_1}, \ldots, \frac{A_p}{x_p}\right\}, \tag{17}$$

where $A_i > 0$. If we set $y_i = \log x_i$ as in Example 2.7, we recover the form (16). The monotonicity of the logarithm implies that ranks of the x_i and y_i are unchanged. Eq. (17) in the case $k = 1$ is the subject of extensive conjectures of Ladas [9] and Grove and Ladas [7]. In the following, we extend Ladas's max-type conjectures to the context of general rank-type equations. We state them in additive form (16), though they are easily translated to the multiplicative form (17).

Definition. The solution $\{x_n\}$ of a difference equation is called *eventually periodic with period p* if there exists an integer $N > 0$ such that $x_{n+p} = x_n$ for all $n \geqslant N$.

Conjecture 4.2. *Consider the difference equation (16) where the $b_1, \ldots, b_p$ are ordered as*

$$b_{i_1} > \cdots > b_{i_{k-1}} > b_{i_k} > b_{i_{k+1}} > \cdots > b_{i_p},$$

that is, b_{i_k} is the kth largest of distinct b_i. Then all solutions are eventually periodic with period $2i_k$. (The prime period may be a divisor of $2i_k$.)

For example, the case $p = 3$, $k = 2$ concerns the equation

$$x_n = \text{median}\{-x_{n-1} + b_1, -x_{n-2} + b_2, -x_{n-3} + b_3\}. \tag{18}$$

The conjecture holds that for any initial conditions, the solution is eventually periodic with period

$$\begin{array}{lll} 2 & \text{if } b_2 < b_1 < b_3 & \text{or } b_3 < b_1 < b_2, \\ 4 & \text{if } b_1 < b_2 < b_3 & \text{or } b_3 < b_2 < b_1, \\ 6 & \text{if } b_1 < b_3 < b_2 & \text{or } b_2 < b_3 < b_1. \end{array}$$

The requirement in Conjecture 4.2 that b_{i_k} is nonrepeating is important. If this requirement is lifted, although eventual periodicity is still expected, the formula for the period is more complicated. Continuing the case $p = 3$, $k = 2$ we have:

Proposition 4.3. *Uncountably many solutions of the Eq. (18) exist with prime periods*

$$\begin{array}{ll} 2 & \textit{if } b_1 = b_3 \neq b_2, \\ 3 & \textit{if } b_1 = b_2 \neq b_3, \\ 4 & \textit{if } b_1 = b_2 = b_3, \\ 5 & \textit{if } b_2 = b_3 \neq b_1. \end{array}$$

Proof. It is easily checked that the following sequences satisfy the difference Eq. (18).

Case 1: $b_1 = b_3 \neq b_2$. For any β, define

$$x_1 = \frac{b_1}{2} + \beta,$$
$$x_2 = \frac{b_1}{2} - \beta.$$

Then $x_1, x_2, x_1, x_2, \ldots$ is a solution.

Case 2: $b_1 = b_2 \neq b_3$. Set

$$x_1 = \frac{b_1}{2} + \beta,$$
$$x_2 = \frac{b_1}{2} + \beta,$$
$$x_3 = \frac{b_1}{2} - \beta,$$

where β is any number between *0* and $(b_3 - b_1)/2$. Then $x_1, x_2, x_3, x_1, x_2, x_3, \ldots$ is a solution.

Case 3: $b_1 = b_2 = b_3$. For any β, set

$$x_1 = \frac{b_1}{2} + \beta,$$
$$x_2 = \frac{b_1}{2} + \beta,$$
$$x_3 = \frac{b_1}{2} - \beta,$$
$$x_4 = \frac{b_1}{2} - \beta.$$

Then $x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, \ldots$ is a solution.

Case 4: $b_2 = b_3 \neq b_1$. Define

$$x_1 = \frac{b_1}{2} + \beta,$$
$$x_2 = \frac{b_1}{2} + \beta,$$
$$x_3 = \frac{b_1}{2} + \beta,$$
$$x_4 = \frac{b_1}{2} - \beta,$$
$$x_5 = \frac{b_1}{2} - \beta,$$

where β is any number between 0 and $(b_1 - b_2)/2$. Then $x_1, x_2, x_3, x_4, x_5, x_1, x_2, x_3, x_4, x_5, \ldots$ is a solution, completing the proof. □

We conjecture that the periodic solutions found above represent all possible periods for the $p = 3$, $k = 2$ rank-type equation. More precisely, we propose the following:

Conjecture 4.4. *Consider the difference equation (18). Then all solutions are eventually periodic, with period*

$$\begin{array}{ll}
2 & \textit{if } b_2 < b_1 < b_3 \quad \textit{or } b_3 < b_1 < b_2, \\
4 & \textit{if } b_1 < b_2 < b_3 \quad \textit{or } b_3 < b_2 < b_1, \\
6 & \textit{if } b_1 < b_3 < b_2 \quad \textit{or } b_2 < b_3 < b_1, \\
2 & \textit{if } b_1 = b_3 \neq b_2, \\
3 & \textit{if } b_1 = b_2 \neq b_3, \\
4 & \textit{if } b_1 = b_2 = b_3, \\
5 & \textit{if } b_2 = b_3 \neq b_1.
\end{array}$$

See [19] for a proof of analogous results for the $p = 3$, $k = 1$ case. We expect that similar methods may suffice to prove Conjecture 4.4.

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